Polynomial Optimization

Optimization Day at MPI Leipzig, February 12, 2018
Some Problems to be discussed in the Afternoon

1. Pack n circles of radius r inside a unit square. How large can r be?

2. Find the orthogonal matrix closest to a multiple of the Hilbert matrix

$$\big[\, 1/(i+j-1) \,\big]_{1 \leq i,j \leq n}.$$

3. Maximize the volume of the convex hull of k points on the unit sphere $\mathbb{S}^2$.

4. Pick a random polynomial of degree d in n variables, and maximize your polynomial over the cube $[-1,1]^n$. Try various values of d and n.

5. Pick a random polynomial of degree d in n variables, and maximize that polynomial over $\{-100, \ldots, 100\}^n$. Try various values of d and n.

6. Find the largest regular m-gon inside a regular n-gon.

7. Does there exist an $n \times n$-matrix that is both Hankel and orthogonal?

8. Consider a Gaussian random vector $X = (X_1, X_2, X_3, X_4)$ whose entries satisfy $X_1 \perp\!\!\!\perp X_2$, $X_3 \perp\!\!\!\perp X_4$, $X_1 \perp\!\!\!\perp X_3 \,|\, \{X_2, X_4\}$, and $X_2 \perp\!\!\!\perp X_4 \,|\, \{X_1, X_3\}$. What is the maximum correlation between X_1 and X_3?

9. Consider the cone of vectors (a,b,c,d,e) in $\mathbb{R}^5$ such that the univariate polynomial $ax^4 + bx^3 + cx^2 + dx + e$ is nonnegative. Describe this cone by a Boolean combination of polynomial inequalities in 5 unknowns.

10. Let $M_k \subset \mathbb{R}^{m\times n}$ be the set of $m \times n$ matrices of rank $\leq k$. Let $X = UV^T \in M_k$ with $U \in \mathbb{R}^{m\times \ell}$ and $V \in \mathbb{R}^{n\times \ell}$ having rank ℓ. Compute the tangent cone of M_k at X. **True or false**: If X is a relative local minimum of a function $f\colon M_k \to \mathbb{R}$ then $\mathrm{rank}(X) = k$ or $\nabla f(X) = 0$?

11. Pick explicit random instances of U, V, k in the previous problem. Determine a semialgebraic description of the corresponding tangent cone.

12. For polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$, we consider the *Rayleigh difference*

$$\Delta_{ij}(p) \; = \; \frac{\partial p}{\partial x_i} \cdot \frac{\partial p}{\partial x_j} - p \cdot \frac{\partial^2 p}{\partial x_i \partial x_j}, \qquad 1 \leq i, j \leq n.$$

Let $e_{k,n}$ be the elementary symmetric polynomial of degree k in n variables. Is $\Delta_{ij}(e_{k,n})$ a sum of squares? Next consider the polynomial

$$V \; = \; e_{4,8} - (x_1x_2x_3x_4 + x_1x_2x_5x_6 + x_1x_2x_7x_8 + x_3x_4x_5x_6 + x_3x_4x_7x_8).$$

For which values of i, j is $\Delta_{ij}(V)$ nonnegative? .... a sum of squares?

13. How to maximize a linear function over the lattice points inside a spectrahedron? On which preprint server can one find the article *"A Framework for Solving Mixed-Integer Semidefinite Programs"*? Why ?

14. Try the SONC Method on two of the problems above. How does it do?