Chow Exercises

Galois extensions and Covering spaces

Exercise 1. Calculate the fundamental groups of $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{U}(n)$...
Hint: consider the action $\mathrm{SO}_n(\mathbb{R})$ on the n-sphere. What is the fiber of the morphism $g \mapsto g.p$, $p = (1, 0, \ldots, 0)$? Use the long exact sequence of homotopy groups, $\pi_2(\mathbb{S}^n, s) = 1$ for all $n \geq 3$, and the cover $\mathrm{SU}(2) \longrightarrow \mathrm{SO}_3(\mathbb{R})$.

Exercise 2. Calculate the splitting field of $x^5 - 2$ over $\mathbb{Q}$. What is the Galois group?

Exercise 3. Let $\pi : X \longrightarrow Y$ be a covering space. Show that X is Hausdorff, if Y is. Is the converse true?

Exercise 4. True or false: if $\pi : X \longrightarrow Y$ is a local homeomorphism, then it is a covering space. Prove the assertion under the additional assumption that Y is path connected and that both X and Y are compact Hausdorff spaces. Can you relax this assumption?

Exercise 5. Let $p \in \mathbb{C}[t]$ be a nonconstant polynomial and $B := \{z \in \mathbb{C} \mid p'(z) = 0\}$. Show that

$$p : \mathbb{C} \setminus p^{-1}(p(B)) \longrightarrow \mathbb{C} \setminus p(B)$$

is a covering space.
Show that for $p(t) = 2t^3 - 3t^2$ this map is not a Galois cover. Give an algebraic and a topological proof. Calculate the Galois closure. What is the Galois group?

Exercise 6. Let $K \subset L$ be a field extension with $d = \dim_K L$. Describe the d-fold tensor product

$$L \otimes_K L \otimes_K \ldots \otimes_K L$$

in case $K \subset L$ is

1. $\mathbb{R} \subset \mathbb{C}$
2. $\mathbb{Q} \subset \mathbb{Q}[\sqrt[3]{2}]$
3. $\mathbb{F}_2(t^2) \subset \mathbb{F}_2(t)$

Exercise 7. Consider the "eight" $Y := \mathbb{S}^1 \bigvee \mathbb{S}^1$ and the two covering spaces depicted below. Are these Galois covers? Can you calculate the Deck group?