EXERCISES FOR "WHAT IS p-ADIC GEOMETRY?"

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Exercise 1.

(1) *Eisenstein irreducibility criterion for polynomials over* $\mathbb{Q}_p$. Let
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$
be a polynomial with p-adic integer coefficients. Show that f is irreducible over $\mathbb{Q}_p$ if a_n is not divisible by p, the coefficients $a_{n-1}, \ldots, a_1$ are divisible by p, and the constant therm a_0 is divisible by p but not p^2.

(2) Show that there exist finite extensions of arbitrary degree of $\mathbb{Q}_p$.

See Exercise 2 of Section 2 of Christoph's exercises for more on Eisenstein polynomials over non-archimedean field.

Exercise 2. Let K be a complete, algebraically closed, non-archimedean valued field, and let $D_r(a)$ be the closed disk $D_r(a) = \{x \in K \mid |x - a| \leq r\}$. Define
$$\begin{array}{cccc} ||\cdot||_{D_r(a)} : & K[T] & \to & \mathbb{R}_{\geq 0} \\ & f & \mapsto & \sup_{x \in D_r(a)} |f(x)| \end{array}$$

(1) Given $f = \sum_i^n c_i T^i$, show that $||f||_{D_1(0)} = \max |a_i|$. This is called the *Gauss norm*.

(2) Show that $||\cdot||_{D_r(a)}$defines a multiplicative seminorm on $K[T]$.

Exercise 3. Let K be a complete, algebraically closed, non-archimedean valued field. Given a positive real number R, let
$$K\langle R^{-1}T\rangle = \Big\{ f = \sum_{k=0}^{\infty} c_k T^k \in K[[T]] \, : \, \lim_{k \to \infty} R^k |c_k| = 0 \Big\},$$
be the ring of formal power series with radius of convergence at least R. It is complete with respect to the norm $||f||_R := \max |c_k| R^k$. We define $\mathcal{D}_0(R)$ as the set of multiplicative seminorms on $K\langle R^{-1}T\rangle$ extending the absolute value on K. We equip it with the weakest topology for which $x \mapsto ||f||_x$ is continuous for all $f \in K\langle R^{-1}T\rangle$.

(1) Explain that if $0 < R' < R$, then $\mathcal{D}_0(R')$ can be seen as a subspace of $\mathcal{D}_o(R)$.

(2) Prove that
$$\mathbb{A}_K^{1,\mathrm{Berk}} \cong \bigcup_{R > 0} \mathcal{D}_R(0).$$

Exercise 4. A valued field K is *spherically complete* if every descending nested sequence of closed disks has nonempty intersection. Prove that the completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}_p}$ is not spherically complete.