EXERCISES FOR RAMIFICATION AND PERFECTOID FIELDS

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1. RAMIFICATION

(1) [CF67, Ch I] Let F be a non-archimedian complete discretelty valued field with perfect residue field. All extensions are supposed to be seperable. Show, that there is a correspondence between.

$$\{\text{unramified extensions of } F\} \quad \longleftrightarrow \quad \{\text{extensions of } \kappa(F)\}$$

- Let E/F is unramified. There exists an element $\alpha \in \mathcal{O}_E$ such that $\kappa(E) = \kappa(F)[\overline{\alpha}]$, where $\overline{\alpha}$ denotes the reduction mod ϖ_F. If f is the minimal polynomial of α over F, then $\mathcal{O}_E = \mathcal{O}_F[\alpha]$, $E = F[\alpha]$ and $\overline{f}$ is seperable and irreducible in $\kappa(F)[T]$.
- Let g be a monic polynomial in $\mathcal{O}_F[T]$ such that $\overline{g}$ is seperable and irreducible in $\kappa(F)[T]$. If β is a root of g then $E = F[\beta]$ is unramified over F and $\kappa(E) = \kappa(F)[\beta]$.

(2) [CF67, Ch I] A seperable, nonzero polynomial $f(X) = b_n X^n + b_{n-1} X^{n-1} + \ldots + b_0 \in F[X]$ is called *Eisenstein polynomial*, if $v_F(b_n) = 0$, $v_F(b_i) \geq 1$ for $1 \leq i < n$ and $v_F(b_0) = 1$. Show, that Eisenstein polynomials correspond to totally ramified extensions:

$$\{\text{totally ramified extensions of } F\} \quad \longleftrightarrow \quad \{\text{roots of Eisenstein polynomials}\}$$

- An Eisenstein polynomial f is irreducible
- If α is a root of f, then $E = F[\alpha]$ is totally ramified over F and $v_E(\alpha) = 1$.
- If E/F is totally ramified and $v_E(\beta) = 1$, then the minimal polynomial of β over F is Eisenstein, $\mathcal{O}_E = \mathcal{O}_F[\beta]$ and $E = F[\beta]$.

2. PERFECTOID FIELDS AND ALMOST MATHEMATICS

(1) Show that the completion of $\mathbb{Q}_p(\mu_{p^\infty})$ is a perfectoid field, where μ_{p^n} is the group of p^n-th roots of unity.
- Consider the extension $\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p$. It is a splitting field, what is the corresponding minimal polynomial?
- Show, that the ramification degree is $p^{n-1}(p-1)$.

(2) [Sta18, Tag 02MN] Construction of the Serre quotient category
- Let $\mathcal{A}$ be an abelian category. A nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *Serre subcategory* if given an exact sequence

$$A \to B \to C$$

with $A, C \in \mathrm{Ob}(\mathcal{C})$, then also $B \in \mathrm{Ob}(\mathcal{C})$.
- There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact, essentially surjective functor $\mathcal{F} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ with kernel $\mathcal{C}$
- Construct $\mathcal{A}/\mathcal{C}$ as the localized category $S^{-1}\mathcal{A}$ where

$$S = \{f \in \mathrm{Arrow}(\mathcal{A}) \mid \ker(f), \mathrm{coker}(f) \in \mathrm{Ob}(\mathcal{C})\}.$$

Convince yourself, that the name *quotient category* makes sense.

- Let A be an integral domain, $(A - \mathrm{Mod})$ the category of A-modules and $\mathcal{T}$ the serre subcategory of torsion modules. Then there is a caononical equivalence of categories

$$(A - \mathrm{Mod})/\mathcal{T} \to (Q(A) - \mathrm{Vect})$$

where the category on the right is the category of $Q(A)$-vectorspaces.

(3) [GR03, 2.2.6] Let K be a perfectoid field. Show that $(\mathcal{O}_K^a - \mathrm{mod})$ has the structure of an abelian tensor category.

- The corresponding objects are defined in such a way, that they are compatible with the functor $\mathcal{F} : \cdot \mapsto \cdot^a$.
- Show, that for M, N $\mathcal{O}_K$-Modules, we have

$$\mathrm{Hom}_{\mathcal{O}_K^a}(M^a, N^a) = \mathrm{Hom}_{\mathcal{O}_K}(\mathfrak{p}_K \otimes M, N)$$

References

[CF67] J. W. S. Cassels and A. Froehlich. *Algebraic Number Theory*. 1967.

[GR03] O. Gabber and L. Ramero. *Almost Ring Theory*. 2003.

[Sta18] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2018.