EXERCISES ON THE THEOREM OF FONTAINE-WINTENBERGER

1. Warm-up Questions

1.1. **"Fermat's Little Theorem".** Let R be a ring, and let $t \in R$ be an element such that $p \in (T)$. Given $a, b \in R$ with $a \equiv b \mod t$, show that $a^{p^n} \equiv b^{p^n} \mod t^{n+1}$ for all $n \geq 0$.

1.2. **Krasner's Lemma.**

(1) Let K be a non-archimedean field and let $\overline{K}$ be a separable closure of K. Given an element $\alpha \in \overline{K}$, denote its Galois conjugates by $\alpha_2, \ldots, \alpha_n$. Then show that if an element $\beta \in \overline{K}$ is such that
$$|\alpha - \beta| < |\alpha - \alpha_i|$$
for $i = 2, \ldots, n$, then $K(\alpha) \subseteq K(\beta)$.

(2) Show how this is used to conclude the proof of the theorem of Fontaine–Wintenberger.

1.3. Let $\mathcal{O}_K$ be the valuation ring of a complete and algebraically closed non-archimedean field. Let $P(T) = \sum_{i=1}^n a_i T^i \in \mathcal{O}_K[T]$ be a polynomial such that $n \geq 1$ and such that $a_0, a_i \in \mathcal{O}_K^\times$ for at least one $1 \leq i \leq n$. Then show that P vanishes at a unit of $\mathcal{O}_K$. (Hint: This can be done either using Newton polygons or via an algebraic argument.)

1.4. Let $f : R \to S$ be a map of characteristic p rings that is surjective with nilpotent kernel. Show that $\lim_\phi R \simeq \lim_\phi S$. Show that the same result holds if f factors a power of Frobenius on either ring.

2. The Tilting Correspondence

2.1. **Examples.** Compute explicitly the tilts of the following rings:

(1) $\mathbb{Z}_p$
(2) $\mathbb{F}_p[t]$
(3) $\mathbb{F}_p((t^{1/p^\infty}))$
(4) $\widehat{\mathbb{Q}_p[p^{1/p^\infty}]}$
(5) $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$
(6) A finite type algebra R over an algebraically closed field k

What happens to the rank of the rings under the tilt?

2.2. Let R be a p-adically complete ring. Show the following:

(1) If R is a domain, then so is its tilt $R^\flat$.
(2) If R is a valuation ring, then so is its tilt $R^\flat$.[1]

2.3. Let R be a ring that is p-adically complete. Show that the projection map $R \to R/p$ induces a bijection of the multiplicative monoids
$$\lim_{x \mapsto x^p} R \to R^\flat.$$
(Hint: Use "Fermat's Little Theorem" in the warm-up questions.)

[1] In fact, the valuation on $R^\flat$ is obtained from the valuation $|\cdot| : R \to \Gamma \cup \{0\}$ by the composition $R^\flat \xrightarrow{\sharp} R \xrightarrow{|\cdot|} \Gamma \cup \{0\}$.

3. Tilting Perfectoid Fields

In this section K will always denote a perfectoid field with corresponding ring of integers K°.

3.1. Check that the following topologies on $K^{\circ\flat}$ are equivalent:

(1) The inverse limit topology via $K^{\circ\flat} \simeq \lim_\phi K^{\circ\flat}/\pi$.
(2) The inverse limit topology via $K^{\circ\flat} \simeq \lim_{x\mapsto x^p} K^\circ$.
(3) The inverse limit topology via $K^{\circ\flat} \simeq \lim_\phi K^\circ/p$, where the topology on K°/p is the one induced from K[2].

3.2. **Tilting Perfectoid Fields.** Here we show step-by-step that the tilt of a perfectoid field will be a perfectoid field. We fix K to be a perfectoid field with a chosen pseudouniformizer π with $|p| \leq |\pi| < 1$, so $p \in (\pi)$.

(1) Verify the commutativity of the following diagram:

$$
\begin{tikzcd}
\lim_{x\mapsto x^p} K^\circ \arrow[r, "\mathrm{pr}"] \arrow[d, "\simeq"] & K^\circ \arrow[d, "\mathrm{red}"] \\
K^{\circ\flat} := \lim_{x\mapsto x^p} K^\circ/p \arrow[r, "\mathrm{pr}"] \arrow[d, "\simeq"] & K^\circ/p \arrow[d, "\mathrm{red}"] \\
\lim_{x\mapsto x^p} K^\circ/\pi \arrow[r, "\mathrm{pr}"] & K^\circ/\pi
\end{tikzcd}
.
$$

(Hint: Use Exercises 1.4 and 2.3.)

(2) Show there exists an element $t \in K^{\circ\flat}$ such that $|t^\sharp| = |\pi|$. Moreover, show that t maps to 0 in K°/π and that this gives an isomorphism $K^{\flat\circ}/t \simeq K^\circ/\pi$. (Hint: Choose an element $f \in K^\circ$ such that $|f|^p = |\pi|$, and analyze a lift of $f \in K^\circ/\pi$ in $K^{\circ\flat}$.)
(3) Show that with t as above that $K^{\circ\flat}$ is t-adically complete, and that the t-adic topology coincides with the given topology.
(4) Conclude that $K^{\circ\flat}$ is a valuation ring and $K^\flat := K^{\circ\flat}[\frac{1}{t}]$ is a perfect (and thus perfectoid) field such that the value groups of K and $K^\flat$ are canonically identified. (Hint: Recall the conclusion of Exercise 2.2.)

3.3. **Algebraically Closed Perfectoid Fields.** Here we show that if $K^\flat$ is algebraically closed, then its untilt K is algebraically closed. The argument is done by inductively constructing a sequence $\{x_n \in K^\circ\}$ such that for each n the following two conditions are satisfied

- $|x_{n+1} - x_n| \leq |p|^{\frac{n}{d}}$,
- $|P(x_n)| \leq |p|^n$ for an (arbitrary) monic polynomial P of degree $d \geq 1$ with coefficients in K°,

where by the first condition we conclude that $\{x_n\}$ converges to some element $x \in K^\circ$, and by the second condition that $|P(x)| = 0$, and thus $P(x) = 0$, showing that K is algebraically closed.

We take $x_0 = 0$ and assume by induction that we have constructed $x_0, \ldots, x_n$ satisfying the above two properties.

[2] This topology is discrete when K has characteristic zero, but is not when K has characteristic p.

(1) Write $P(T + x_n) = \sum_{i=0}^{d} b_i T^i$ and show the quantity
$$c = \min\left\{ \left|\frac{b_0}{b_j}\right|^{\frac{1}{j}} : j > 0, b_j \neq 0 \right\}$$
satisfies $c = |u|$ for some unit $u \in K^\circ$. Use this to show that there exists $i > 0$ such that $\frac{b_i}{b_0} \cdot u^i \in K^\circ$ is a unit.

(2) Let $Q(T) \in K^{\circ\flat}[T]$ be any polynomial lifting $\sum_{i=0}^{d} \frac{b_i}{b_0} u^i T^i \in K^\circ/p[T]$, where we use the identification $K^{\circ\flat}/t \simeq K^\circ/p$ to construct the lift. Show there exists a unit $y \in K^{\circ\flat}$ such that $Q(y) = 0$. (Hint: Use Exercise 1.3)

(3) Check that the element $x_{n+1} = x_n + u \cdot y^\sharp \in K^\circ$ satisfies the two itemized conditions.

4. Witt Vectors and $\mathbb{A}_{\mathrm{inf}}$

4.1. **Witt Vectors of Characteristic p Rings.** Fix R to be a ring of characteristic p.

(1) Given $x \in R$, show that the Teichmüller lift $[x] := \lim_{n\to\infty} (\overline{x^{1/p^n}})^{p^n} \in W(R)$ is well-defined, independent of the choices of representatives $\overline{x^{1/p^n}} \in W(R)$ of each $x^{1/p^n} \in R$.

(2) Show that for any $x \in W(R)$, there is a unique sequence of elements $\{c_n\}_{n\geq 0} \in R$ such that
$$x = \sum_{n\geq 0} [c_n] p^n.$$
Such a sum is called a *Teichmüller representation* of $x \in W(R)$.

(3) Verify the following universal property: for any p-adically complete ring A, reduction modulo p induces a bijection:
$$\mathrm{Hom}(W(R), A) \xrightarrow{\sim} \mathrm{Hom}(R, A/pA).$$

4.2. Verify[3], by using the universal properties of $\mathbb{A}_{\mathrm{inf}}$, that for $L^\flat$ a finite Galois extension of the perfectoid field $K^\flat$,
$$L^{\flat\sharp} \simeq (W(L^{\flat\circ}) \otimes_{\mathbb{A}_{\mathrm{inf}}(K^\circ)} K^\circ)[\frac{1}{\pi}].$$

4.3. **Étale Infinitesimal Lifting property.** Let R be any ring.

(1) Show that for any surjective map $\widetilde{R} \to R$ with a nilpotent kernel, base change induces an equivalence between the category of étale R-algebras and the category of étale $\widetilde{R}$-algebras.

(2) Show that if R is equipped with a non-zero divisor $f \in R$, then reduction modulo f induces an equivalence between the category of f-adically complete and f-torsionfree R-algebras S with $R/f \to S/f$ étale and the category of étale R/f-algebras.

(3) Verify that for $L^\flat$ a finite Galois extension of the perfectoid field $K^\flat$, the field $L^{\flat\sharp} = (W(L^{\flat\circ}) \otimes_{\mathbb{A}_{\mathrm{inf}}(K^\circ)} K^\circ)[\frac{1}{\pi}]$ constructed in the lecture is a finite Galois extension of K (and of the same degree!).

[3] In particular, this verifies that the untilt of $L^\flat$ as a field extension of $K^\flat$ via Fontaine's construction of the untilt coincides with the untilt operator on fields.