# Exercises for Summer School 

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The first two exercises are on the Niyogi-Smale-Weinberger Theorem.

1. Let $X, W \subseteq \mathbb{R}^{n}$ be nonempty and closed. Provide examples showing that

$$
d_{H}(X, W)<\epsilon<\tau(W)
$$

are necessary conditions for $W$ being a deformation retract of $U(X, \epsilon)$. (Here $d_{H}$ denotes the Hausdorff distance and $\tau(W)$ denotes the reach of $W$.)
2. Let $W \subseteq \mathbb{R}^{n}$ be a compact set of positive reach and assume $\epsilon<\frac{1}{2} \tau(W)$. Assume $x_{1}, \ldots, x_{N}$ are chosen independently at random with respect to a probability distribution $\mu$ on $W$. Let $0<\delta<1$ be a confidence level.
Claim. The sampled set $X=\left\{x_{1}, \ldots x_{N}\right\}$ satisfies $d_{H}(X, W)<\epsilon / 3$ with probability at least $1-\delta$, when

$$
N \geq \frac{1}{\alpha}\left(\log \ell+\log \frac{1}{\delta}\right)
$$

where

$$
\alpha:=\inf _{w \in W} \mu(W \cap B(w, \epsilon / 12)), \quad \ell:=\left\lfloor\frac{\mu(W)}{\alpha}\right\rfloor .
$$

Proceed as follows.
a. Show there exists a covering $W \subseteq B\left(p_{1}, \epsilon / 6\right) \cup \cdots \cup B\left(p_{\ell}, \epsilon / 6\right)$ with $p_{i} \in W$.
b. For $d_{H}(X, W)<\epsilon / 3$ it is enough to have

$$
\forall i \leq \ell \exists j \leq N \quad d\left(p_{i}, x_{j}\right)<\epsilon / 6 .
$$

Bound the probability that this fails.
3. Invariance of Weyl inner product. The Weyl inner product on the space of homogeneous polynomials of degree $d$ in $n+1$ variables is given by

$$
\langle f, g\rangle:=\sum_{\alpha}\binom{d}{\alpha} f_{\alpha} g_{\alpha}, \quad \text { where } \quad f=\sum_{\alpha}\binom{d}{\alpha} f_{\alpha} X^{\alpha}, g=\sum_{\alpha}\binom{d}{\alpha} g_{\alpha} X^{\alpha}
$$

Here $X^{\alpha}=X_{0}^{\alpha_{0}} \cdots X_{n}^{\alpha_{n}}$ and the sums are over all $\alpha \in \mathbb{N}^{n+1}$ such that $\sum_{i} \alpha_{i}=d$. The Weyl inner product is invariant under orthogonal transformations: $\langle f \circ u, g \circ u\rangle=\langle f, g\rangle$ for $u \in O(n+1)$.

There is an easy an argument for this: first note that

$$
\langle S, T\rangle:=\sum_{0 \leq i_{1}, \ldots \leq n} S_{i_{1} \cdots i_{d}} T_{i_{1} \cdots i_{d}} \quad \text { where } \quad S=\left[S_{i_{1} \cdots i_{d}}\right], \quad T=\left[T_{i_{1} \cdots i_{d}}\right]
$$

defines an orthogonal invariant inner product on the space $\mathbb{R}^{n+1} \otimes \ldots \otimes \mathbb{R}^{n+1}$ of real tensors of order $d$. Now view $f, g$ as the symmetric tensors $S, T$ with the entries $S_{i_{1} \cdots i_{d}}=f_{\alpha}$, $T_{i_{1} \cdots i_{d}}=g_{\alpha}$, where $\alpha_{k}=\#\left\{i \mid i=i_{k}\right\}$. Then $\langle S, T\rangle=\langle f, g\rangle$.
4. Let $f$ be homogeneous of degree $d$ and $u \in O(n+1)$ be a rotation in a plane by the angle $\theta$. Show that

$$
\|f-f \circ u\| \leq d\|f\| \theta
$$

Hint: Consider a more general complex statement where $u \in U(n+1)$ is unitary. By unitary invariance one may assume that $u$ is diagonal. Then calculate.
5. Lipschitz properties of $\kappa^{-1}$. Recall the Condition Number Theorem states that for $f \in \mathcal{H}_{d}[q]$ and $x \in \mathbb{S}^{n}:$

$$
\kappa(f, x)=\frac{\|f\|}{d\left(f, \Sigma_{x}\right)},
$$

where $\Sigma_{x}$ consists of the systems which have $x$ as a multiple zero.
Deduce from this the following.
a. (Very easy) Fix $x \in \mathbb{S}^{n}$. The map $f \mapsto \kappa^{-1}(f, x)$ is 1 -Lipschitz continuous, when we restrict to systems of norm 1 :

$$
\left|\kappa^{-1}(f, x)-\kappa^{-1}(g, x)\right| \leq\|f-g\| .
$$

b. Let $f \in \mathcal{H}_{\underline{d}}[q]$ and $u \in O(n+1)$ be a rotation in a plane by the angle $\theta$. Then

$$
\|f-f \circ u\| \leq D\|f\| \theta
$$

where $D=\max _{i} d_{i}$. Hint: One can assume $q=1$ without loss of generality. Use exercise 4 .
c. Fix $f \in \mathcal{H}_{\underline{d}}[q]$. Use (b) to show that the map

$$
\mathbb{S}^{n} \rightarrow \mathbb{R}, x \mapsto \kappa^{-1}(f, x)
$$

is $D$-Lipschitz continuous, with respect to the Riemannian distance (angle) on $\mathbb{S}^{n}$.

