Discrete Volume Computations for Polyhedra

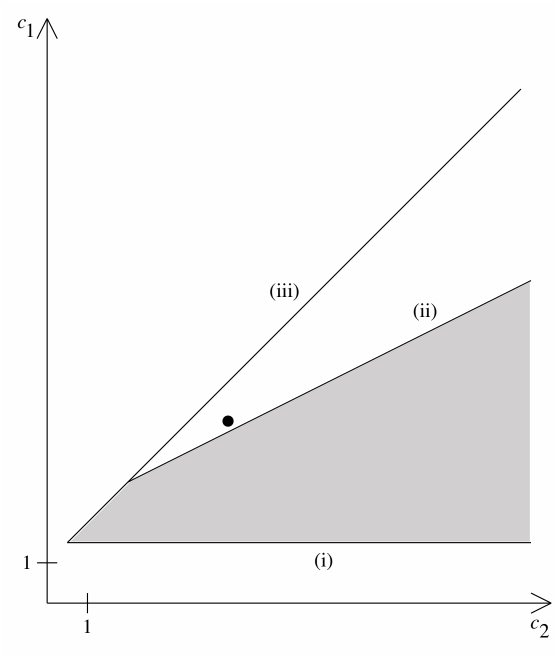
A Few Appetizers

I. Ehrhart Polynomials

(1) Pick five points in $\mathbb{Z}^3$ and let $\mathcal{P}$ be their convex hull (in $\mathbb{R}^3$). Compute the hyperplane description of $\mathcal{P}$.

(2) Compute the Ehrhart polynomial of $\mathcal{P}$.

(3) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2 t^2 + c_1 t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



(4) Give the corresponding classification picture of degree-3 Ehrhart polynomials. (This is open.)

II. Generating Functions

(1) Show that a sequence $f(n)$ is given by a polynomial of degree $\le d$ if and only if

$$\sum_{n \ge 0} f(n)\, z^n \;=\; \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\le d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

(2) Give an example of a polynomial $f(n)$ with (some) negative coefficients whose corresponding generating function numerator polynomial $h(z)$ has only positive coefficients.

(3) For a lattice polytope $\mathcal{P}$, the numerator of the generating function is the *h^*-polynomial* of $\mathcal{P}$. Give a non-unimodal example of an h^*-polynomial.

(4) Now let $\mathcal{P} = \left\{ \mathbf{x} \in [0,1]^d : x_1 + x_2 + \cdots + x_d = k \right\}$, for your favorite integers $1 \le k < d$. (This is the *(d,k)-hypersimplex*.) Prove that the h^*-polynomial of $\mathcal{P}$ is unimodal. (This is open.)

III. Polynomial Method 101

(1) Let $\mathcal{B}_n$ be the set of all doubly-stochastic $n \times n$ matrices. Show that the vertices of $\mathcal{B}_n$ are the $n \times n$ permutation matrices.

(2) Compute the Ehrhart polynomial of $\mathcal{B}_n$ for the first few positive integers n. (This is open for $n \ge 10$.)

(3) Let $G = (V, E)$ be a connected bipartite graph with $|V| > 2$. An edge labeling of G is *antimagic* if all the sums of the labels of the edges incident to each vertex distinct). Let $A_G(n)$ be the number of antimagic labelings of G where each label is an integer between 1 and n. Prove that $A_G(n)$ is a polynomial.

(4) Assuming that $A_G(0) = 0$ (you may think about why this holds), show that $A_G(|E|) > 0$.

(5) Now let $\widetilde{A}_G(n)$ be defined as above, except that each edge gets assigned a *distinct* label. Prove that $\widetilde{A}_G(|E|) > 0$. (This is the *Antimagic Graph Conjecture*; it is open already for trees.)

IV. Betke–McMullen–Stapledon

(1) For any polynomial $h(z)$ and any $d \in \mathbb{Z}_{>0}$, show there exist unique polynomials $a(z)$ and $b(z)$ such that

$$h(z) = a(z) + z\, b(z) \qquad \text{where} \qquad a(z) = z^d\, a(\tfrac{1}{z}) \qquad \text{and} \qquad b(z) = z^{d-1}\, b(\tfrac{1}{z})\,.$$

(There are many variations of this; e.g., we could leave out the z factor in front of $b(z)$.)

(2) Derive inequalities for the coefficients of $h(z)$ if we know that both $a(z)$ and $b(z)$ have only nonnegative coefficients.

(3) A linear inequality $a_0 x_0 + a_1 x_1 + \cdots + a_d x_d \ge 0$ is *balanced* if $a_0 + a_1 + \cdots + a_d = 0$. Find a complete set of balanced inequalities for the coefficients of every h^*-polynomial of degree d for the first positive integers d. (This is open for $d \ge 6$.)

(4) In the Ehrhart setting, the nonnegativity of $h^*(z)$ was proved in 1980, whereas the nonnegativitiy of the accompanying polynomials $a(z)$ and $b(z)$ (which is, naturally, stronger) has been known only for 10 years. Try this same setup for your favorite (combinatorial) polynomial with nonnegative coefficients.