Convex Geometry Exercises

Max Planck Institute for Mathematics in the Sciences

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1 Convexity and Optimization

Exercises:

1. Fix a linear functional $a \in (\mathbb{R}^n)^\star$ and a set $\mathcal{S} \subset \mathbb{R}^n$. Show that

$$\sup\{\langle a, x\rangle \mid x \in \mathcal{S}\} = \sup\{\langle a, x\rangle \mid x \in \operatorname{conv}(\mathcal{S})\}.$$

2. Consider a convex body $\mathcal{C} \subset \mathbb{R}^n$ and a linear functional $a \in (\mathbb{R}^n)^\star$. Show that the maximizers of the following optimization problem:

$$\text{maximize } \langle a, x\rangle \quad \text{subject to} \quad x \in \mathcal{C}$$

   constitute an exposed face of $\mathcal{C}$.

3. Consider the following spectrahedron:

$$\mathcal{S} = \left\{ x \mid \sum_{i=1}^{n} A^{(i)} x_i \preceq B \right\}$$

   Here $A^{(1)}, \ldots, A^{(n)}, B \in \mathbb{S}^k$. If the matrices $A^{(1)}, \ldots, A^{(n)}, B$ are simultaneously diagonalizable, then show that $\mathcal{S}$ is a polyhedron. (This result generalizes the observation from the lectures that if $A^{(1)}, \ldots, A^{(n)}, B$ are diagonal, then $\mathcal{S}$ is a polyhedron.)

4. Draw a labelled figure describing the following spectrahedron:

$$\mathcal{S} = \left\{ x \in \mathbb{R}^3 \mid \begin{pmatrix} 1+x_1 & x_2 & 0 & 0 \\ x_2 & 1-x_1 & 0 & 0 \\ 0 & 0 & 1+x_3 & 0 \\ 0 & 0 & 0 & 1-x_3 \end{pmatrix} \succeq 0 \right\}.$$

5. Show that every point on the boundary of the Euclidean ball in $\mathbb{R}^n$ is an extreme point of the Euclidean ball.

6. This problem concerns the faces of the spectraplex:

$$\text{Spec}_k = \{X \in \mathbb{S}^k \mid X \succeq 0,\ \text{tr}(X) = 1\}.$$

   (a) Fix any subspace $\mathcal{V} \subseteq \mathbb{R}^k$. Show that the set $\{M \in \mathbb{S}^k \mid \text{col-space}(M) \subseteq \mathcal{V}\} \cap \text{Spec}_k$ is a face of Spec_k. Here col-space$(\cdot)$ refers to the column space.

   (b) Are the faces from the previous part exposed?

7. Describe the face lattices of the ℓ_1 and ℓ_∞ norm balls in $\mathbb{R}^2$ and in $\mathbb{R}^3$. Comment on any notable properties.

8. Consider a bounded polyhedron $\mathcal{P} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$:

$$\mathcal{P} = \{x \mid \langle a^{(i)}, x\rangle \leq 1,\ i = 1, \ldots, k\}.$$

   The goal of this problem is to show that $\mathcal{P}^\circ = \text{conv}\{a^{(i)},\ i = 1, \ldots, k\}$.

   (a) First, show that $\text{conv}\{a^{(i)},\ i = 1, \ldots, k\} \subseteq \mathcal{P}^\circ$.

   (b) To show the opposite inclusion, we'll appeal to the separation theorem from convex analysis. Specifically, a consequence of this theorem is that if a point $x \notin \mathcal{C}$ for a compact convex set $\mathcal{C} \subset \mathbb{R}^n$, then there exists a linear functional $c \in (\mathbb{R}^n)^\star$ such that:

$$\langle c, x\rangle > \sup\{\langle c, y\rangle \mid y \in \mathcal{C}\}.$$

   Using this result (without proof), show that $\mathcal{P}^\circ \subseteq \text{conv}\{a^{(i)},\ i = 1, \ldots, k\}$. Combined with the previous part, you can conclude that $\mathcal{P}^\circ = \text{conv}\{a^{(i)},\ i = 1, \ldots, k\}$.

   (c) Interpret this result in the context of the ℓ_1 and ℓ_∞ norm balls in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Research problems:

1. Consider the following spectrahedron:

$$\mathcal{S} = \left\{ x \mid \sum_{i=1}^{n} A^{(i)} x_i \preceq B \right\}$$

   Here $A^{(1)}, \ldots, A^{(n)}, B \in \mathbb{S}^k$. Provide conditions (beyond simultaneous diagonalizability) on the matrices $A^{(1)}, \ldots, A^{(n)}, B$ such that $\mathcal{S}$ is a polyhedron.

2. If the symmetric matrices in the description of a spectrahedron are diagonal matrices, then we obtain a polyhedron. Suppose instead we replace those symmetric matrices with block diagonal symmetric matrices where each block is 2×2. What kinds of convex objects do we get? In $\mathbb{R}^3$? In general?

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