Convex Geometry Exercises

Max Planck Institute for Mathematics in the Sciences

July 2021

1 Polyhedra

1. (a) Find the vertices of the following $\mathcal{H}$-polyhedra:
   - i. $P = \{(x, y, z) \mid x + y \leq 2, y + z \leq 4, x + z \leq 3, -2x - y \leq 3, -y - 2z \leq 3, -2x - z \leq 2\}$
   - ii. $P = \{(x, y) \mid x \geq 0, y \geq 0, y - x \leq 2, x + y \leq 8, x + 2y \leq 10, x \leq 4\}$.

   (b) How can we identify vertices from the structure of A for $P = \{x : Ax \leq b\}$?

2. Prove that a point v in a polytope $P \subset \mathbb{R}^d$ cannot be written as a convex combination two distinct points of P if and only if it is a zero-dimensional face of P, that is, $v = P \cap \{x : c^\top x = \gamma\}$ for some $c \in \mathbb{R}^d, \gamma \in \mathbb{R}$.

3. Let $A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m$. Prove the following versions of Farkas' Lemma:

   (a) The system $Ax = b$ has a nonnegative solution if and only if there is no vector y satisfying $y^\top A \geq 0$ and $y^\top b < 0$.

   (b) Prove that there exists a vector $x \neq 0$ with $x \geq 0$ such that $Ax = 0$ if and only if there is no vector y satisfying $y^\top A > 0$. (Gordan's theorem (Gordan [1873]).)

4. Prove Carathéodory's theorem: If $z \in \operatorname{conv}(X)$, then there exist affinely independent vectors $x_1, \ldots, x_m \in X$ such that $z \in \operatorname{conv}\{x_1, \ldots, x_m\}$. (Note: this means any point in a d-polytope is a convex combination of at most $d + 1$ vertices.)
   Recall $x_1, \ldots, x_m$ are affinely independent if $\sum_{i=1}^m \lambda_i x_i = 0$ and $\sum_{i=1}^m \lambda_i = 0$ only for λ_i all zero. (There was a slight error in the way this way stated in lecture.)

5. Prove the the intersection of two faces of a polytope P is also a face of P.
   Let F be a face of P. Prove that the faces of F are exactly the faces of P contained in F.

6. Show that every subset of vertices of the standard $(n - 1)$-simplex form a face.

7. (a) Prove the following facts about the poset of faces of a d-polytope.

   i. It is a lattice.
   ii. It is graded of rank $d+1$.
   iii. It is atomic and coatomic.

   (b) Which properties hold for the faces of a non-polyhedral convex body?

   (c) Pick two polytopes. Compute the face lattice of their join, product, and direct sum.

8. Choose your favorite polytope $P = \text{conv}\{x_1, \dots, x_n\} \subset \mathbb{R}^d$. Can you write down a polytope $Q = \text{conv}\{y_1, \dots, y_n\} \subset \mathbb{R}^d$ so that Q is not an affine transformation of P? (That is $Q \neq TP + t$ for some invertible $T \in \mathbb{R}^{d\times d}$ and $t \in \mathbb{R}^d$

9. Let $K \subset \mathbb{R}^d$ and let $K^\circ = \{y \in \mathbb{R}^d : y^\top x \leq 1, \text{ for all } x \in K\}$ be its polar. Show

   (a) K° is closed and convex.

   (b) $K \subset L \Rightarrow L^\circ \subset K^\circ$

   (c) $K \subseteq K^{\circ\circ}$. If, in addition, If K is closed and convex with $0 \in K$ then $K^{\circ\circ} = K$.

10. Pick your favorite 3-polytope.

   (a) What polytope do you get when you apply the affine projection

$$\pi(x) = \begin{bmatrix} \sqrt{3} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   (b) A *projective transformation* on $\mathbb{R}^d$ is a map of the form

$$\phi(x) = \frac{Ax + b}{c^\top x + \gamma}$$

   for $A \in \mathbb{R}^{d\times d}, b, c \in \mathbb{R}^d, \gamma \in \mathbb{R}$ with $\det \begin{bmatrix} A & c^\top \\ b & \gamma \end{bmatrix} \neq 0$. What happens to your polytope under some projective transformations of $\mathbb{R}^3$?

11. Recall, the extension complexity of a polytope P is the minimum number of facets of a polytope Q which projects onto P. Give an upper bound on the extension complexity of a polytope with n vertices.

References

[1] Alexander Barvinok. *A course in convexity*, volume 54. American Mathematical Soc., 2002.

[2] Grigoriy Blekherman, Pablo A. Parrilo, and Rekha R. Thomas, editors. *Semidefinite optimization and convex algebraic geometry*, volume 13 of *MOS-SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013.

[3] Venkat Chandrasekaran and James Saunderson. Terracini Convexity, 2020. arXiv preprint: arxiv.org/abs/2010.00805.

[4] Richard J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, second edition, 2006.

[5] Pierre Lairez, Marc Mezzarobba, and Mohab Safey El Din. Computing the volume of compact semi-algebraic sets. In *Proceedings of the 2019 on International Symposium on Symbolic and Algebraic Computation*, ISSAC '19, page 259–266, New York, NY, USA, 2019. Association for Computing Machinery.

[6] Daniel Plaumann, Rainer Sinn, and Jannik Lennart Wesner. Families of faces and the normal cycle of a convex semi-algebraic set, 2021.

[7] Motakuri Ramana and A. J. Goldman. Some geometric results in semidefinite programming. *J. Global Optim.*, 7(1):33–50, 1995.

[8] Kristian Ranestad and Bernd Sturmfels. The convex hull of a variety. In Petter Brändén, Mikael Passare, and Mihai Putinar, editors, *Notions of Positivity and the Geometry of Polynomials*, pages 331–344. Springer Verlag, Basel, 2011.

[9] Kristian Ranestad and Bernd Sturmfels. On the convex hull of a space curve. *Advances in Geometry*, 12(1):157–178, 2012.

[10] Philipp Rostalski and Bernd Sturmfels. Dualities in convex algebraic geometry. *Rendiconti di Mathematica*, 30:285–327, 2010.

[11] Raman Sanyal, Frank Sottile, and Bernd Sturmfels. Orbitopes. *Mathematika*, 57(2):275–314, 2011.

[12] Rolf Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2014.

[13] Rainer Sinn. *Algebraic Boundaries of Convex Semi-Algebraic Sets*. PhD thesis, Universität Konstanz, Konstanz, 2014.

[14] Rainer Sinn. Algebraic boundaries of convex semi-algebraic sets. *Research in the Mathematical Sciences*, 2(1):3, 2015.