Convex Geometry Exercises

Max Planck Institute for Mathematics in the Sciences

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1 Non-polyhedral Convex Sets

1. Consider the union of the unit disk in $\mathbb{R}^2$ centered at the origin and the convex hull of $\{(-1,1),(-1,-1),(0,-1),(0,1)\}$. Is this a basic semialgebraic set?

2. Spectrahedra are always closed. What about projected spectrahedra?

3. Is the intersection of two spectrahedra a spectrahedron?

4. Every face of a spectrahedron is an exposed face.

   Given a point x in a spectrahedron, denote by $F(x)$ the unique face that contains x in its relative interior. Let $S = \{x \in \mathbb{R}^n : A(x) = A_0 + A_1x_1 + \cdots + A_nx_n \succeq 0\}$ be a spectrahedron.

   Theorem 1 *For $\bar{x}$ the affine hull of $F(\bar{x})$ is*

   $$aff(F(\bar{x})) = \{x \in \mathbb{R}^n : null(A(\bar{x}) \subset null(A(x))\}$$

   where $null(X)$ is the null space of the matrix X and

   $$F(\bar{x}) = aff(F(\bar{x})) \cap S.$$

   (a) Assume $\bar{x} = 0$ and assume that $A(x)$ is in the form,

   $$A(x) = \begin{bmatrix} \tilde{A}_0 + \tilde{A}(x) & B(x)^T \\ B(x) & C(x) \end{bmatrix}$$

   where $C(x)$ is $k \times k$ and $\tilde{A}(x), B(x)$, and $C(x)$ are matrices with linear entries in x, i.e. no constant terms. Further assume $\tilde{A}_0 \succ 0$. Prove that $aff(F(\bar{x})) = \{x \in \mathbb{R}^n : C(x) = 0, B(x) = 0\}$.

   (b) Define the vector a such that $a_i = \mathrm{Tr}(C_i)$. Prove that

   $$F(0) = \{x \in S : a^T x = 0\}.$$

(c) Why can we make the assumptions in part (a) without loss of generality?

5. Show that $p(x, y) = 1 - x^4 - y^4$ is not a real zero polynomial at the origin.

6. Prove that the PSD cone is Terracini convex. Note: there was a mistake in the video lecture when defining the normal cone of a convex cone C. It is defined as $\mathcal{N}_C(x) = \{l \in C^\circ : l(x) = 0\}$.

Toward Research

1. Compute the projection body of 3-elliptope,

$$\{(x, y, z) : \begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \succeq 0\}.$$

What can we say about the projection body of the elliptope in the general?

2. Define the "f-vector" of zonoids/spectrahedra/general convex bodies.

3. A simplicial polytope is one where every proper face is a simplex. Consider in the non-polyhedral setting a closed pointed convex cone for which every proper face is Terracini convex. We say such convex cones are *boundary Terracini convex*. Classify all boundary Terracini convex cones. (Some examples can be found in the paper on Terracini convexity)

4. What is Oscar? https://oscar.computeralgebra.de
Write an Oscar package for your favorite non-polyhedral convex body.

5. Skim through the table of contents in Ziegler's book Lectures on Polytopes and pick a chapter or section title. How can properties/statements/theorems/questions in this section be restated or asked for non-polyhedral convex bodies.

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