Convex Geometry Exercises

Max Planck Institute for Mathematics in the Sciences

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1 Algebra and Convexity

1. Guess the picture! Characterise and/or give equations for the pictures that appear in the slides behind the titles of the sections.

2. Slide 10/56: the discotope! What are the hidden terms in the equation of the violet surface? In other words: parametrize and compute the equation of the violet surface.

3. Prove the "basic properties" of duality in slide 17/56 (for the third point: assume that the origin is inside K_1 and K_2).

4. Let K be a convex body in the plane different than a polygon. Suppose there is a vertex of K where two nonlinear components intersect transversely. Then the dual convex body K° is not a basic closed semialgebraic set.

5. There was a small mistake in Chiara's lecture, concerning the algebraic boundary and convex cones. Spot it and correct it!

6. Let $C = \{q_1 = 0\} \cap \{q_2 = 0\}$, where q_i is a quadric in $\mathbb{R}^3$. Consider the pencil of quadrics $q_1 + tq_2$. In slide 43/56 the following fact was stated: "the edge surface (of C) is the union of the four singular quadrics (of the pencil)". Prove it, assuming that the matrices associated to q_1, q_2 are simultaneously diagonalizable.
   How does this relate to the research problem 2 of the first exercise session?

7. Try the algorithm mentioned in slide 46/56 for the curve

$$C = \left\{ \Big(\cos(\theta), \sin(\theta) + \cos(2\theta), \sin(2\theta) \Big) \mid \theta \in [0, 2\pi] \right\}.$$

8. What is an efficient way to compute the volume of a semialgebraic convex body, with high (i.e. at least 100 digits) precision? Key word: *Picard-Fuchs equations.*
   Compute the volume of the spectrahedron in slide 9/56. You may want to start with the volume of a 2–dimensional section of such convex body.

Toward research:

1. Consider a 3–dimensional semialgebraic version of an ice cream cone, with one strawberry scoop. Give equations for its algebraic boundary. Can you compute the dual body? Can you compute the normal cycle? Can you describe its patches?
   What if you add one lemon scoop? Make this object convex and compute again algebraic boundary, dual body, normal cycle and patches.

2. Let $K \subset \mathbb{R}^n$ be a semialgebraic convex body. Find the hyperplane through the origin H that maximises the volume of $K \cap H$.

3. Let $K \subset \mathbb{R}^n$ be a semialgebraic convex body. Is its projection body ΠK again semialgebraic?
   If so, what is the relation between $\partial_a K$ and $\partial_a \Pi K$?
   If not, give a counterexample.

4. Write an Oscar package that computes $X^{[k]}$ for a given variety X.

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