Twenty Facets of Convex Bodies

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What follows is a list of twenty research questions, one for each facet of the icosahedron. They are meant to connect classical convexity with algebraic geometry and combinatorics, and lead to new tools at the interface of convex optimization and numerical nonlinear algebra.

(1) Consider convex bodies that are obtained as the convex hull of a trigonometric curve of degree d in $\mathbb{R}^3$? What is the maximum number of triangle facets of such a body?

(2) Pick three random ellipsoids in $\mathbb{R}^3$. Compute their convex hull and its dual. What are the combinatorial types of convex bodies that arise? Which arrangements of *patches*?

(3) Let P be a polytope in $\mathbb{R}^n$ and B the unit ball. Is the Minkowski sum $P + B$ a *basic semialgebraic* set? Can you find a description of $P + B$ by polynomial inequalities?

(4) Let S be a spectrahedron in $\mathbb{R}^3$ and B the unit ball. The coefficients of the polynomial function $x \mapsto \mathrm{vol}(S+xB)$ are the *quermassintegrals* of S. Are these numbers algebraic? If not, what about holonomic tools? Start with the example when S is the elliptope.

(5) Study the *projection bodies* of spectrahedra in $\mathbb{R}^3$. What are their patches?

(6) *Quartic spectrahedra* in $\mathbb{R}^3$ have been classified. Compute the *normal cycle* for each type. Extend this to spectrahedra of higher degree and to spectrahedra in $\mathbb{R}^4$.

(7) Revisit known results about the facial structure of *Grassmann orbitopes*. What happens when we restrict these convex bodies to matroid strata? More precisely, consider the matroid strata in the unit sphere in Plücker space, and study their convex hulls.

(8) Study combinatorial types of spectrahedra defined by several symmetric 2×2 matrices. For instance, which convex bodies arise from intersecting four quadratic cones in $\mathbb{R}^3$?

(9) Can all ranks in the *Pataki range* be realized simultaneously on one spectrahedron?

(10) Consider a linear projection of the PSD cone. The image is a semialgebraic convex cone that need not be closed. How to characterize this image by polynomial conditions?

(11) Consider the set $C = \{x \in \mathbb{R}^n : f(x) \leq 1\}$ where $f(x)$ is a polynomial in n real variables. How do you test whether C is compact and convex? Implement your algorithm.

(12) Give examples of polynomials f in $n = 3$ variables of degree four such that the set C in (11) is a zonoid. Do these f form a semialgebraic subset in $\mathbb{R}[x, y, z]_{\leq 4} \simeq \mathbb{R}^{35}$?

(13) Consider a trigonometric curve in $\mathbb{R}^3$ such as $\{(\cos(\theta), \sin(2\theta), \cos(3\theta)) : \theta \in [0, 2\pi]\}$. Compute the *Vitale zonoid* of the curve. Draw some pictures of such Vitale zonoids.

(14) Study compact surfaces that are intersections of two quadrics in $\mathbb{R}^4$. Compute their convex hulls. How is this related to obtaining the surface by blowing five points in $\mathbb{C}^2$?

(15) Let $\mathcal{K} = \{K_1, \ldots, K_m\}$ be a set of centrally symmetric convex bodies in $\mathbb{R}^n$. Each of these defines a norm $|| \cdot ||_{K_i}$ on $\mathbb{R}^n$, which has K_i for its unit ball. Consider the map

$$\psi_{\mathcal{K}} : \mathbb{R}^n \to \Delta_{m-1} : x \mapsto \frac{1}{\sum_{i=1}^m ||x||_{K_i}} \cdot \big(||x||_{K_1}, ||x||_{K_2}, \ldots, ||x||_{K_m} \big).$$

Study the image and the fibers of the map $\psi_{\mathcal{K}}$. Identify the branch loci of $\psi_{\mathcal{K}}$.

(16) The *width* of a $\mathbb{Q}$-defined semi-algebraic convex body K is an algebraic number, and it can be computed by polynomial optimization (how?). Give an algorithm to decide whether K is *reduced*, i.e. every convex body strictly contained in K has smaller width.

(17) Consider the convex body defined by a *hyperbolic polynomial* f. Find an algorithm whose input is the polynomial f and which decides whether the body is a zonoid.

(18) Study *Voronoi cells* of algebraic curves in 4-dimensional Euclidean space. What features do we see in their boundary? Find exact representations. Draw many pictures.

(19) A *Hankel tensor* is a homogeneous polynomial that is a linear combination of powers $(x_0 + ux_1 + u^2x_2 + \cdots + u^n x_n)^d$. Is every nonnegative Hankel tensor a sum of squares?

(20) Given a finite set of matrices in $\mathbb{R}^{m \times n}$, is their *rank-one convex hull* a semi-algebraic set?