Special Session on Sums of Squares

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(1) Prove that every face F of $\mathbb{S}_+^n$ of dimension $\binom{r+1}{2}$ contains a matrix of rank r in its relative interior.

(2) Show that for every face F of $\mathbb{S}_+^n$ of dimension $d < \binom{r+1}{2} \leq \binom{n+1}{2}$ there is a face F' of $\mathbb{S}_+^n$ of dimension $\binom{r+1}{2}$ such that $F \subset F'$.

(3) Show that the dual cone of $\mathbb{S}_+^n$ is again $\mathbb{S}_+^n$ (using the inner product $\langle A, B\rangle = \mathrm{tr}(AB)$ to identify the space of real symmetric matrices with its dual vector space).

(4) Show that the group action $\mathrm{O}(n) \times \mathbb{R}_{sym}^{n\times n} \to \mathbb{R}_{sym}^{n\times n}$, $(O, A) \mapsto OAO^\top$ is *isometric* (again, the chosen inner product is the same as in the previous problem).

(5) Fix $d = 3$. Pick a polynomial $p \in \mathbb{R}[t]_{\leq 2d}$ and find all extreme points on the Gram spectrahedron of this polynomial. What are their ranks? What do you expect the general behavior to be (say for a random polynomial)? Can you answer the above questions for $d = 4$ or $d = 5$?

(6) What is the Pataki range for the dimensions arising for Gram spectrahedra of univariate polynomials of degree $2d$?

(7) On Bernd's list "Twenty Facets of Convex Bodies", problem 9 asks for a spectrahedron with extreme points of all ranks in the appropriate Pataki range. Can you use Gram spectrahedra of univariate polynomials of degrees 8, 10, or maybe 12 to find such examples for the appropriate Pataki ranges?

(8) Can you find examples of "Gram" spectrahedra that have extreme points of all ranks in the Pataki range for Gram maps $\mathcal{G}\colon \mathbb{R}_{sym}^{n\times n} \to V$ that are coordinate projections (in other words, the kernels of these maps are coordinate subspaces)? Try $n = 4$ and projections with a kernel of dimension at least 3. Or $n = 5$ and kernels of dimension at least 6.

(9) Show that the universal SO(2)-orbitopes are spectrahedra. (*Hint:* Google Fejer-Riesz Theorem)
Coordinate projections of these convex bodies appear again in problem 1 on Bernd's list.

(10) Again, fix the inner product $\langle A, B\rangle = \mathrm{tr}(AB)$ on $\mathbb{R}_{sym}^{n\times n}$. Consider the convex cone $C = L \cap \mathbb{S}_+^n$ for a linear space $L \subset \mathbb{R}_{sym}^{n\times n}$. Show that the dual cone of C is the orthogonal projection $\pi_{L^\perp}(\mathbb{S}_+^n)$ of $\mathbb{S}_+^n$ away from the orthogonal complement of L. Can you use this fact to describe a relationship between Hankel spectrahedra and the Gram map?

(11) Show that the convex cone generated by the matrix

$$M = \begin{pmatrix} \frac{3}{4} & -1 & 0 & 0 & \frac{1}{4} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ \frac{1}{4} & 0 & 0 & -1 & \frac{3}{4} \end{pmatrix}$$

is an extreme ray of the spectrahdron of all 5×5 matrices where the entries a_{ij} are 0 for $(i, j) \in \{(1, 3), (1, 4), (2, 4), (2, 5), (3, 5)\}$. (In other words, only the diagonal entries, the entries on the first off-diagonal, and the entry in the top right corner are allowed to be non-zero.) Can you generalize this to larger matrix size?

(12) What is a *Gorenstein ideal* and what does it have to do with Hankel spectrahedra?

(13) Can you prove Conjecture 7.3 in the paper "Nonnegative Polynomials and Sums of Squares" by Grigoriy Blekherman? (arXiv:1010.3465v2)