

Exercise 1. Let H be a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Recall that a *Gaussian isonormal process* on H is a centered Gaussian process $W: H \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_H \forall g, h \in H.$$

- i. Show that W is linear.
- ii. Show that for given H and a sufficiently rich probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a Gaussian isonormal process always exists
- iii. Let $W: L^2(\mathbb{T}^d) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a Gaussian isonormal process. Show that there exists a unique *white noise*, i.e. a random variable $\xi: \Omega \rightarrow \mathcal{S}'^1$ such that for all $f \in \mathcal{S}$, $\xi(f)$ is centered Gaussian and $\mathbb{E}[\xi(f)\xi(g)] = \langle f, g \rangle_{L^2(\mathbb{T}^d)}$, with the property that

$$W(f) = \xi(f), \mathbb{P} - a.s.$$

Hint: Show that $\mathbb{E}[\sum_{k \in \mathbb{Z}^d} |W(e_k)|^2 (1 + |k|^2)^{-s}] < \infty$ for some s .

- iv. Conversely, show that for such a given white noise $\xi: \Omega \rightarrow \mathcal{S}'$ one can construct a unique Gaussian isonormal process $W: L^2(\mathbb{T}^d) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $W(f) = \xi(f), \mathbb{P} - a.s.$

Exercise 2. Let $d=2$. A *Gaussian free field* (with mass 1) is a random variable $\eta: \Omega \rightarrow \mathcal{S}'$ such that for $f \in \mathcal{S}$, $\eta(f)$ is centered Gaussian with covariance

$$\mathbb{E}[\eta(f)\eta(g)] = \langle (1 - \Delta)^{-1/2} f, (1 - \Delta)^{-1/2} g \rangle_{L^2(\mathbb{T}^2)}.$$

- i. Let $\xi: \Omega \rightarrow \mathcal{S}'(\mathbb{T}^2)$ be a white noise. Show that $\eta = (1 - \Delta)^{-1/2} \xi$ is a Gaussian free field. Using your solution from exercise 1, deduce that $\mathbb{P} - a.s.$,

$$\eta \in H^{0-}(\mathbb{T}^2) = \bigcap_{\epsilon > 0} H^{-\epsilon}(\mathbb{T}^2).$$

- ii. Show that $\mathbb{P} - a.s.$,

$$\eta \in B_{\infty, \infty}^{0-}(\mathbb{T}^2) = \bigcap_{\epsilon > 0} B_{\infty, \infty}^{-\epsilon}(\mathbb{T}^2)$$

Hint: You can use the embedding $\|\cdot\|_{B_{\infty, \infty}^{\alpha-2/p}} \lesssim \|\cdot\|_{B_{p, \infty}^{\alpha}} \leq \|\cdot\|_{B_{p, p}^{\alpha}}$ for any $p \geq 1$.

Exercise 3. Let $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be smooth and let ρ be a probability density on \mathbb{R}^d for the probability measure $\mu(dx) = \rho(x) dx$.

- i. Show that the following statements are equivalent

- a) μ is “infinitesimally invariant” for the dynamics $x'(t) = b(x_t)$, i.e.

$$\int \mathcal{L}_b f \mu = 0, f \in C_c^\infty(\mathbb{R}^d),$$

where $\mathcal{L}_b f = b \cdot \nabla f$.

- b) \mathcal{L}_b is antisymmetric w.r.t. $L^2(\mu)$

- ii. Suppose b is divergence-free. For a quantity $E: \mathbb{R}^d \rightarrow \mathbb{R}^{\geq 0}$ such that $\int e^{-E(x)} dx < \infty$ show that with $\rho(x) = (\int e^{-E(y)} dy)^{-1} e^{-E(x)}$, μ is “infinitesimally invariant” iff E is a conserved quantity for the dynamics $x'(t) = b(x_t)$.

- iii. On a formal level, show that the following non-linearities on \mathbb{T}^d are divergence-free in terms of Fourier coefficients $(\hat{u}(k))_{k \in \mathbb{T}^d}$ (i.e. as “infinite-dimensional vector-fields”) and show that they preserve the respective quantities E .

- a) $u: \mathbb{T} \rightarrow \mathbb{R}, b(u) = \partial_x u^n$, where $n \in \mathbb{N}$. $E(u) = \|u\|_{L^2(\mathbb{T})}^2$ (general Burgers non-linearity)

- b) $h: \mathbb{T}^2 \rightarrow \mathbb{R}, b(h) = (\partial_1 h)^2 - (\partial_2 h)^2$. $E(h) = \|\nabla h\|_{L^2(\mathbb{T}^2)}^2$ (cf. Anisotropic KPZ equation)

- c) $\omega: \mathbb{T}^2 \rightarrow \mathbb{R}, b(\omega) = (\nabla^\perp (-\Delta)^{-\gamma} \omega) \cdot \nabla \omega$, where $\gamma \in [1, \frac{1}{2}]$. $E(\omega) = \|\omega\|_{L^2(\mathbb{T}^2)}^2$, $\bar{E}(\omega) = \|(-\Delta)^{-\gamma/2} \omega\|_{L^2(\mathbb{T}^2)}^2$ (cf. Navier-Stokes/Euler/surface-quasigeostrophic equations)

- d) $u: \mathbb{T}^2 \rightarrow \mathbb{R}, b(u) = \eta \cdot \nabla u$, where $\eta \in \mathcal{S}'(\mathbb{T}^2, \mathbb{R}^2)$ is divergence free. $E(u) = \|u\|_{L^2(\mathbb{T}^2)}^2$ (Stochastic transport term)

1. The topology on $\mathcal{S}'(\mathbb{T}^d)$ is defined to be the smallest topology such that all the seminorms $p_f(u) = |u(f)|$ for $f \in \mathcal{S}$ are continuous. Hence $u_n \rightarrow u$ iff $u_n(f) \rightarrow u(f)$ for all \mathcal{S} .