Exercise 1

Let $b\colon \mathbb{R}^d \to \mathbb{R}^d$ be smooth and let ρ be a probability density on $\mathbb{R}^d$ for the probability measure $\mu(dx) = \rho(x)\,dx$.

i. Show that the following statements are equivalent

   a) μ is "infinitesimally invariant" for the dynamics $x'(t) = b(x_t)$, i.e.

   $$\int \mathcal{L}_b f \mu = 0\,,\ f \in C_c^\infty(\mathbb{R}^d),$$

   where $\mathcal{L}_b f = b \cdot \nabla f$.

   b) $\mathcal{L}_b$ is antisymmetric w.r.t. $L^2(\mu)$

ii. Suppose b is divergence-free. For a quantity $E\colon \mathbb{R}^d \to \mathbb{R}^{\geq 0}$ such that $\int e^{-E(x)} dx < \infty$ show that with $\rho(x) = (\int e^{-E(y)} dy)^{-1}\, e^{-E(x)}$, μ is "infinitesimally invariant" iff E is a conserved quantity for the dynamics $x'(t) = b(x_t)$.

iii. On a formal level, show that the following non-linearities on $\mathbb{T}^d$ are divergence-free in terms of Fourier coefficients $(\hat{u}(k))_{k \in \mathbb{T}^d}$ (i.e. as "infinite-dimensional vector-fields") and show that they preserve the respective quantities E.

   a) $u\colon \mathbb{T} \to \mathbb{R}, b(u) = \partial_x u^n$, where $n \in \mathbb{N}$. $E(u) = \|u\|_{L^2(\mathbb{T})}^2$ (general Burgers non-linearity)

   b) $h\colon \mathbb{T}^2 \to \mathbb{R}, b(h) = (\partial_1 h)^2 - (\partial_2 h)^2$. $E(h) = \|\nabla h\|_{L^2(\mathbb{T}^2)}^2$ (cf. Anisotropic KPZ equation)

   c) $\omega\colon \mathbb{T}^2 \to \mathbb{R}, b(\omega) = (\nabla^\perp (-\Delta)^{-\gamma} \omega) \cdot \nabla \omega$, where $\gamma \in \left[1, \frac{1}{2}\right]$. $E(\omega) = \|\omega\|_{L^2(\mathbb{T})}^2, \bar{E}(\omega) = \|(-\Delta)^{-\gamma/2} \omega\|_{L^2(\mathbb{T}^2)}^2$ (cf. Navier-Stokes/Euler/surface-quasigeostrophic equations)

   d) $u\colon \mathbb{T}^2 \to \mathbb{R}$, $b(u) = \eta \cdot \nabla u$, where is $\eta \in \mathcal{S}'(\mathbb{T}^2, \mathbb{R}^2)$ is divergence free. $E(u) = \|u\|_{L^2(\mathbb{T})}^2$ (Stochastic transport term)

Exercise 2

Let μ be the law of white noise on $H^{-1/2-}(\mathbb{T})$. We define a Gaussian isonormal process W: $L^2(\mathbb{T}, \mathrm{Leb}) \to L^2(\mu)$ (i.e. $\Omega = H^{-1/2-}(\mathbb{T})$) by extending $\tilde{W}: \mathcal{S}(\mathbb{T}) \to L^2(\mu)$

$$\tilde{W}(f)(u) = u(f) \tag{1}$$

by denseness and isometry to the whole space $L^2(\mathbb{T})$. Gaussian analysis (cf. [Nualart, The Malliavin Calculus and related topics, 1995] gives us the following structure on $L^2(\mu)$:

- There exist linear maps $W_n: L^2(\mathbb{T}^n) \to L^2(\mu)$ such that $\mathcal{H}_n = W_n(L^2(\mathbb{T}^n))$ yields an orthogonal decomposition

$$L^2(\mu) = \bigoplus_{n \geq 0} \mathcal{H}_n ,$$

  such that for $\varphi = \sum_{n \geq 0} W_n(\varphi_n)$ one has

$$\|\varphi\|^2_{L^2(\mu)} = \sum_{n \geq 0} n! \|\Pi\varphi_n\|^2_{L^2(\mathbb{T}^n)} ,$$

  where $\Pi\varphi_n := \frac{1}{n!}\sum_{\sigma \in \Sigma_n} \varphi(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is the symmetrization of φ_n.

- Furthermore, denoting the *r-contraction*

$$f \otimes_r g(r_1, \ldots, r_{m+n-2r}) = \int_{\mathbb{T}^r} f(x_1, \ldots, x_r, r_1, \ldots, r_{m-r}) g(x_1, \ldots, x_r, r_{m-r+1}, \ldots, r_{m+n-2r}),$$

  for symmetric $f \in L^2(\mathbb{T}^m)$, $g \in L^2(\mathbb{T}^n)$ we get the multiplication rule

$$W_m(f) W_n(g) = \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} W_{m+n-2r}(f \otimes_r g) . \tag{2}$$

- Lastly, for a *cylinder function* $\varphi(u) = \Phi(u(f_1), ..., u(f_n))$, where (for convenience) $\Phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $f_i \in C^\infty(\mathbb{T})$ one can define the *Malliavin derivative*

$$D_x \varphi(u) = \sum_{i=1}^{n} \partial_i \Phi(u(f_1), \ldots, u(f_n)) f_i(x) . \tag{3}$$

  which has the following representation:

$$D_x W_n(\varphi_n) = n W_{n-1}(\varphi_n(x, \cdot)) . \tag{4}$$

i. Check that on cylinder functions (an in the stationary setting) the contribution of a drift $b(u): \mathcal{S} \to \mathcal{S}$ to the generator of some equation

$$\partial_t u = F(u, \xi) + b(u),$$

is given by

$$\mathcal{G}\varphi(u) = \int_{\mathbb{T}} D_x \varphi\,(u) b(u)(x)\, dx .$$

ii. Using (1) and (2), show that for $b(u) = \partial_x u^2 = 2u\partial_x u$ one has

$$b(u)(x) = 2W_2(\delta_x \otimes \partial\delta_x u)$$

iii. Use the rules (2) and (4) to compute the representation of $(\mathcal{G}W_n(\varphi_n))_m$ (given some n and symmetric $\varphi_n \in L^2(\mathbb{T}^n)$). You should obtain a decomposition

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_- + \mathcal{G}_{-3}\,,$$

such that $\mathcal{G}_+\mathcal{H}_n \subset \mathcal{H}_{n+1}, \mathcal{G}_-\mathcal{H}_n \subset \mathcal{H}_{n-1}$ and $\mathcal{G}_{-3}\mathcal{H}_n \subset \mathcal{H}_{n-3}$. It turns out that

$$\mathcal{G}_{-3} = 0 \quad .$$

Why is this to be expected?