Exercise 1

Let μ be the law of white noise on $H^{-1/2-}(\mathbb{T})$. We define a Gaussian isonormal process W: $L^2(\mathbb{T}, \mathrm{Leb}) \to L^2(\mu)$ (i.e. $\Omega = H^{-1/2-}(\mathbb{T})$) by extending $\tilde{W}: \mathcal{S}(\mathbb{T}) \to L^2(\mu)$

$$\tilde{W}(f)(u) = u(f) \tag{1}$$

by denseness and isometry to the whole space $L^2(\mathbb{T})$. Gaussian analysis (cf. [Nualart, The Malliavin Calculus and related topics, 1995] gives us the following structure on $L^2(\mu)$:

- There exist linear maps $W_n: L^2(\mathbb{T}^n) \to L^2(\mu)$ such that $\mathcal{H}_n = W_n(L^2(\mathbb{T}^n))$ yields an orthogonal decomposition

  $$L^2(\mu) = \bigoplus_{n \geq 0} \mathcal{H}_n ,$$

  such that for $\varphi = \sum_{n \geq 0} W_n(\varphi_n)$ one has

  $$\|\varphi\|_{L^2(\mu)}^2 = \sum_{n \geq 0} n! \|\Pi \varphi_n\|_{L^2(\mathbb{T}^n)}^2 ,$$

  where $\Pi \varphi_n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varphi(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is the symmetrization of φ_n.

- Furthermore, denoting the *r-contraction*

  $$f \otimes_r g(r_1, \ldots, r_{m+n-2r}) = \int_{\mathbb{T}^r} f(x_1, \ldots, x_r, r_1, \ldots, r_{m-r}) g(x_1, \ldots, x_r, r_{m-r+1}, \ldots, r_{m+n-2r}),$$

  for symmetric $f \in L^2(\mathbb{T}^m)$, $g \in L^2(\mathbb{T}^n)$ we get the multiplication rule

  $$W_m(f) W_n(g) = \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} W_{m+n-2r}(f \otimes_r g) . \tag{2}$$

- Lastly, for a *cylinder function* $\varphi(u) = \Phi(u(f_1), ..., u(f_n))$, where (for convenience) $\Phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $f_i \in C^\infty(\mathbb{T})$ one can define the *Malliavin derivative*

  $$D_x \varphi(u) = \sum_{i=1}^{n} \partial_i \Phi(u(f_1), \ldots, u(f_n)) f_i(x) . \tag{3}$$

  which has the following representation:

  $$D_x W_n(\varphi_n) = n W_{n-1}(\varphi_n(x, \cdot)) . \tag{4}$$

i. Check that on cylinder functions the contribution of a drift $b(u): \mathcal{S} \to \mathcal{S}$ to the generator of some equation

$$\partial_t u = F(u, \xi) + b(u),$$

is given by

$$\mathcal{G}\varphi(u) = \int_{\mathbb{T}} D_x \varphi\,(u) b(u)(x)\, dx .$$

ii. Using (1) and (2), show that (under stationarity) for $b(u) = \partial_x u^2 = 2u\partial_x u$ one has

$$b(u)(x) = 2W_2(\delta_x \otimes \partial\delta_x u)$$

iii. Use the rules (2) and (4) to compute the representation of $(\mathcal{G}W_n(\varphi_n))_m$ (given some n and symmetric $\varphi_n \in L^2(\mathbb{T}^n)$). You should obtain a decomposition

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_- + \mathcal{G}_{-3}\,,$$

such that $\mathcal{G}_+\mathcal{H}_n \subset \mathcal{H}_{n+1}, \mathcal{G}_-\mathcal{H}_n \subset \mathcal{H}_{n-1}$ and $\mathcal{G}_{-3}\mathcal{H}_n \subset \mathcal{H}_{n-3}$. It turns out that

$$\mathcal{G}_{-3} = 0 \quad .$$

Why is this to be expected?

Exercise 2

To estimate $\mathcal{G}_\pm$ in terms of $\mathcal{L}_0, \mathcal{N}$, the following estimate is very useful:

Let $C \geq 0, \alpha > 1/2$ and $k \in \mathbb{Z}$ be such that $k^2 + C > 0$. Show that

$$\sum_{p+q=k} \left(\frac{1}{p^2 + q^2 + C}\right)^a = \sum_p \left(\frac{1}{p^2 + (k-p)^2 + C}\right)^a \lesssim \left(\frac{1}{k^2 + C}\right)^{a-1/2}.$$