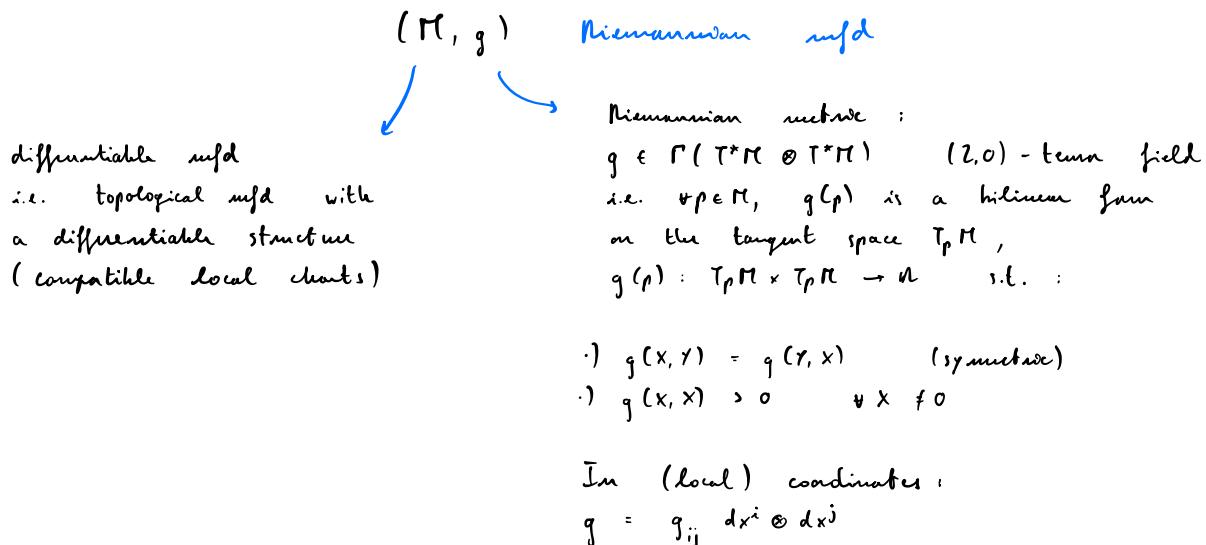


EX CLASS 1 - Ricci curvature, Bochner inequality, and stochastic analysis

Marco Flaum - HCM Bonn - flaum@iam.uni-bonn.de

1) Recalls from Riemannian geometry



A fundamental tool to study Riem. mfds is the **Ricci curvature tensor** ((4,0)-tensor field), which captures the "second derivatives of the metric g ".

In coordinates, it can be seen as the error in the commutation of derivatives (in the sense of Levi-Civita connection).

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = \sum_{\ell=1}^m R_{ijk\ell} X^\ell$$

↗ Einstein summation convention.

A simpler, but still relevant, object is the **Ricci curvature** which is given by the trace (\cong average) of Ricci.

$$\text{Ric}(X, Y) = \sum_{i=1}^m \text{Riem}(X, e_i, Y, e_i)$$

or equivalently $\text{Ric}_{ij} = \text{Riem}_{ij\mu\nu}$.

The Ricci tensor is (as the metric) a $(2,0)$ -tensor ($\Gamma(T^*M \otimes T^*M)$),

i.e. $\forall p \in M, \quad \text{Ric}(p) : T_p M \times T_p M \rightarrow \mathbb{R}$.

- Intuitively :
- 1) Riemann tensor captures the "whole curvature" or "second derivative" of the Riem. mfd.
 - 2) Ricci tensor contains less information (for $n \geq 4$), and is related to the behaviour of the volume form of (M, g) :

$$d\mu_g = \left(1 - \frac{1}{6} R_{jk} x^j x^k + O(|x|^3) \right) d\mu_{(\mathbb{R}^n, \varepsilon)}$$

General philosophy : control of Ricci (from below)
 \downarrow
 strong (topological) conclusions on (M, g) .

Two (classical) such results :

i) Bishop-Gromov's theorem

(M, g) complete Riem. mfd. such that

$$\text{Ric} \geq (n-1)k, \quad \text{i.e. } \text{Ric}(x, x) \geq (n-1)k \underbrace{g(x, x)}_{\text{or ignore this term and deal with unit-vectors.}} \quad \forall x \in TM$$

for some $k \in \mathbb{R}$. Then, $\forall p \in M, \quad r \geq 0$,

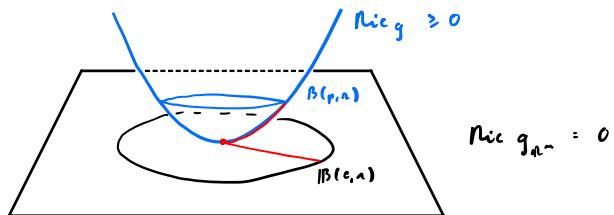
$$\mu_g(B_p(r)) \leq \mu_k(B_o^k(r))$$

measure of (M, g) same quantity associated to the k -space form (mfd of constant curvature k):

geodesic ball in (M, g)

$\left\{ \begin{array}{l} k > 0 \\ k = 0 \\ k < 0 \end{array} \right. : \quad \begin{array}{l} S^n \text{ sphere} \\ \mathbb{R}^n \text{ euclidean space} \\ H^n \text{ hyperbolic space} \end{array}$	$\left\{ \begin{array}{l} k > 0 \\ k = 0 \\ k < 0 \end{array} \right. : \quad \begin{array}{l} S^n \text{ sphere} \\ \mathbb{R}^n \text{ euclidean space} \\ H^n \text{ hyperbolic space} \end{array}$
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Picture :



"If Ricci is bigger, it contains less volume".

ii) Bonnet - Myers' theorem:

(M, g) complete connected Riem. mfd. such that $\text{Ric} \geq (n-1)k$,

for some $k > 0$. Then $d(M) \leq \pi/k$.

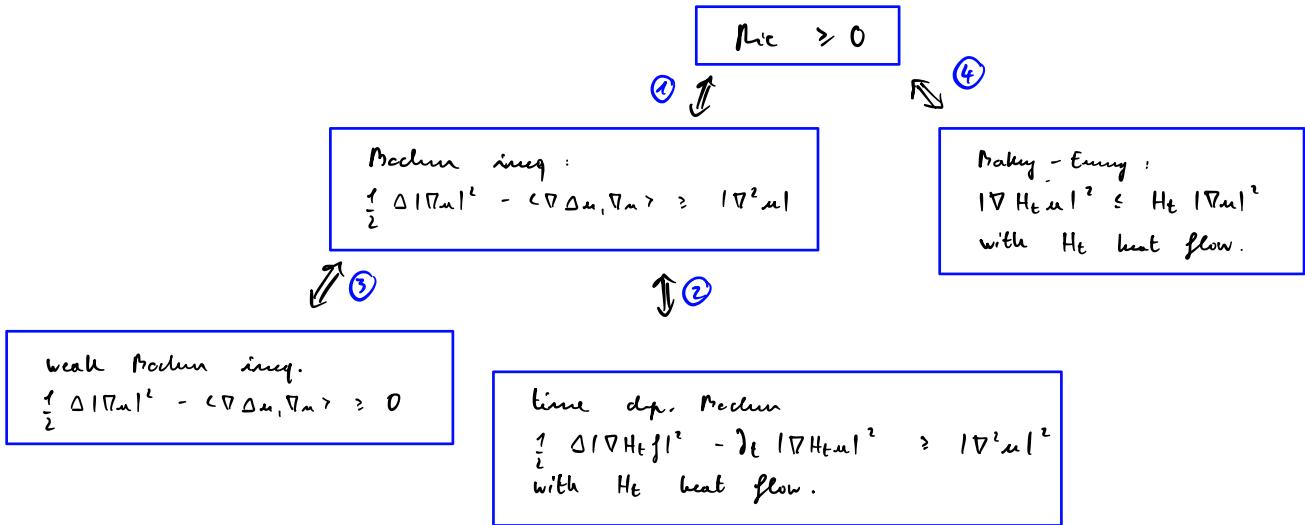
↓
diameter of $M := \sup \{ d(p, q) \mid p, q \in M \}$

In part., (M, g) is compact; in other words a non compact topological mfd does not support a Riem. metric with Ricci bounded from below by some $k > 0$.

Now we are then interested in some characterizations of Ricci bounds from below. We will deal, for the sake of simplicity, with the case $\text{Ric} \geq 0$ ($k = 0$), but the theory can be easily extended to $k \in \mathbb{R}$.

2) Ricci curvature, Bochner inequality, Bakry - Emery estimate

The general scheme is the following :



Lemma (Bochner formula) :

$$\frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla \Delta u, \nabla u \rangle = |\nabla^2 u|^2 + \underset{\substack{\text{Hilbert-Schmidt norm.} \\ \hookrightarrow \text{Hessian of } u}}{\text{Ric}(\nabla u, \nabla u)}$$

$\forall u \in C^2(M)$.

Proof :

$$\begin{aligned}
 \frac{1}{2} \Delta |\nabla u|^2 &= \frac{1}{2} \nabla_i \nabla^i |\nabla u|^2 \\
 &= \frac{1}{2} \nabla_i (\cancel{\nabla^i \nabla_k u \nabla^k u}) \\
 &\quad \cancel{\text{or free change}} \\
 &= \nabla_i \nabla^i \nabla_k u \nabla^k u + \nabla^i \nabla_k u \nabla_i \nabla^k u \\
 &= \nabla_i \nabla_k \nabla^i u \nabla^k u + |\nabla^2 u|^2 \\
 &\quad \cancel{\text{change with error given by Riemann}} \\
 &\quad \text{(in this case Ricci, we have index } i \text{ twice)} \\
 &= (\underbrace{\nabla_k \nabla_i \nabla^i u}_{\Delta u} + \text{Ric}_k^i \nabla^i u) \nabla^k u + |\nabla^2 u|^2 \\
 &= \langle \nabla \Delta u, \nabla u \rangle + \text{Ric}(\nabla u, \nabla u) + |\nabla^2 u|^2
 \end{aligned}$$

□

Proofs of the scheme above:

1) One direction is trivial.

To get the nice bound, combine Modern disp. with Modern formula.

$$\text{Pic}(\nabla u, \nabla u) \geq 0$$

and $\forall X \in T_p M$ we can find $u : \nabla u(p) = X$.

2) Follows from $\partial_t |\nabla H_t u|^2 = \langle \nabla \Delta H_t u, H_t u \rangle$ (exercise).

3) It can be surprising that the weak Modern implies the Modern.

However, proceed as in (1) to get

$$\text{Pic}(\nabla u, \nabla u) + |\nabla^2 u| \geq 0.$$

At this point, note that $\forall X \in T_p M \exists u : \nabla u(p) = X, |\nabla^2 u|(p) = 0$.

To see this, take normal coordinates and compose with the linear function $\tilde{u} : M^m \rightarrow M, \tilde{u}(y) = \langle X, y \rangle_{M^m}$.

4) $L^2 ME \Rightarrow$ weak Modern

$$\varphi(t) = |\nabla H_t f|^2 - H_t |\nabla f|^2$$

$$\varphi(0) = |\nabla f|^2 - |\nabla f|^2 = 0, \quad \varphi(t) \leq 0 \quad \text{by } ME.$$

Then $\varphi'(0) \leq 0$ by h.p., i.e.

$$\varphi'(t) = \partial_t |\nabla H_t f|^2 \Big|_{t=0} - \partial_t (H_t |\nabla f|^2) \Big|_{t=0} \leq 0$$

$$2 \langle \partial_t \nabla H_t f, \nabla H_t f \rangle \Big|_{t=0} - \frac{1}{2} \Delta H_t |\nabla f|^2 \Big|_{t=0} \leq 0$$

$$2 \langle \nabla \frac{1}{2} \Delta f, \nabla f \rangle - \frac{1}{2} \Delta |\nabla f|^2 \leq 0$$

$$\langle \nabla \Delta f, \nabla f \rangle - \frac{1}{2} \Delta |\nabla f|^2 \leq 0$$

$$\text{i.e. } \frac{1}{2} \Delta |\nabla u|^2 \geq \langle \nabla \Delta u, \nabla u \rangle$$

•) weak Pochme $\Rightarrow L^2$ -BE

$$\begin{aligned}\frac{d}{ds} H_s |\nabla H_{t-s} f|^2 &= \frac{1}{2} \Delta H_s |\nabla H_{t-s} f|^2 + 2 H_s \langle \nabla \frac{d}{ds} H_{t-s} f, \nabla H_{t-s} f \rangle \\ &= \frac{1}{2} H_s \Delta |\nabla H_{t-s} f|^2 - H_s \langle \nabla \Delta H_{t-s} f, \nabla H_{t-s} f \rangle \\ &\stackrel{\text{Pochme}}{=} H_s \left(\frac{1}{2} \Delta |\nabla H_{t-s} f|^2 - \langle \nabla \Delta H_{t-s} f, \nabla H_{t-s} f \rangle \right) \\ &\stackrel{\text{max principle}}{\geq} 0\end{aligned}$$

i.e. $g(s) = H_s |\nabla H_{t-s} f|^2$ has $g'(s) \geq 0$, integrating

$$|\nabla H_t f|^2 = g(0) \leq g(t) \leq H_t |\nabla f|^2$$

3) Two sided - Ricci bounds : the general idea

The topic of the mini-course is the extension of such theory to two sided Ricci bounds :

$$|\text{Ric}| \leq k, \text{ i.e. } -k|x|^2 \leq \text{Ric}(x, x) \leq k|x|^2.$$

Note that this is equivalent to $\text{Ric}(x, y) \geq -k|x||y|$

The idea is that we need a Bochner formula with a term $\text{Ric}(x, y)$ instead of $\text{Ric}(x, x)$.

In order to do this, we need to consider functions on the path space $P\mathcal{M}$ instead of just functions on \mathcal{M} .

4) Recalls on path-space analysis

$$PM = \{ \gamma: [c, +\infty) \rightarrow M, \gamma \text{ continuous} \}$$

.) $F: PM \rightarrow \mathbb{R}$ function on PM .

F is a cylinder function if $\exists 0 \leq t_1 < \dots < t_n < +\infty, n \in \mathbb{N}$:

$$F(\gamma) = f(\gamma(t_1), \dots, \gamma(t_n)) \quad \text{for some } f: M^n \rightarrow \mathbb{R}, \forall \gamma \in PM$$

.) Wiener measure.

Fix $x \in M$, we can construct the BM centered at x and the heat kernel $p_t(x, y)$.

The Wiener measure centered at x , P_x , is defined on $\mathcal{B}(PM)$ has marginals given by:

$$\forall 0 \leq t_1 < \dots < t_n < +\infty, n \in \mathbb{N},$$

$$P_x(\gamma(t_1) \in U_1, \dots, \gamma(t_n) \in U_n) =$$

$$= \int_{U_1 \times \dots \times U_n} p_{t_1}(x, y_1) p_{t_2-t_1}(y_1, y_2) \dots p_{t_n-t_{n-1}}(y_{n-1}, y_n) dV(y_1) \dots dV(y_n)$$

Once proven the consistency, we can use Kolmogorov to ensure existence of such a measure.

.) The natural filtration Σ_t is the one induced by ϵ_t .

Equivalently, Σ_t contains the events depending only on $\gamma|_{[c, t]}$.

5) Examples of martingales in PM

i) Let $F \in L^2(\Omega, \mathcal{F}_\infty)$. Then

$$F_t(\gamma) = E[F(\gamma) | \Sigma_t] \quad \text{is a martingale.}$$

Sol:

By the tower property of conditional expectation,

we see,

$$F_s = E[F | \Sigma_s] = E[E[F | \Sigma_t] | \Sigma_s] = E[F_t | \Sigma_s].$$

Note: Conversely, any martingale which is uniformly integrable has such a representation.

ii) Fix $T > 0$, and $f: \mathbb{R} \rightarrow \mathbb{R}$, $f \in C_c^\infty(\mathbb{R})$. Let

$$F_t(\gamma) = \begin{cases} H_{T-t} f(\gamma(t)) & \text{if } t < T, \\ f(\gamma(T)) & \text{if } t \geq T. \end{cases}$$

where H_t is the heat flow.

Then F_t is a martingale and it is induced by

$$F(\gamma) = f(\gamma(T)) \quad \text{in the sense of (i).}$$

Sol:

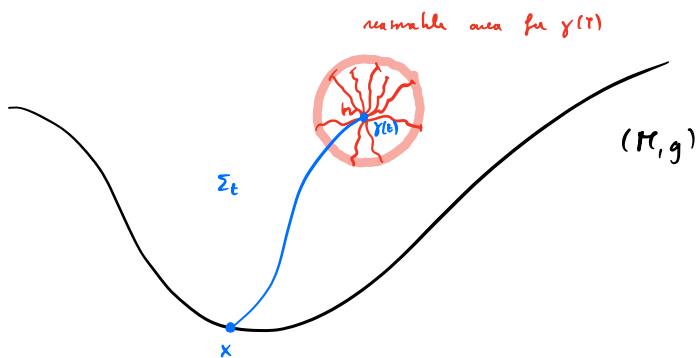
For $t < T$,

$$\begin{aligned} E[F(\gamma) | \Sigma_t] &\stackrel{\text{def of } \Sigma_t}{=} \int_{P_{\gamma(t)}\mathbb{R}} F(\gamma|_{[t, T]} * \gamma') d\Gamma_{\gamma(t)}(\gamma') \\ &= \int_{P_{\gamma(t)}\mathbb{R}} f(\gamma'(\tau-t)) d\Gamma_{\gamma(t)}(\gamma') \end{aligned}$$

$$\begin{aligned}
 & \stackrel{\text{change of var.}}{=} \int_M f(x) d(\epsilon_{\tau-t} \circ \Gamma_{\gamma(t)})(x) \\
 & \stackrel{\text{def of } \Gamma}{=} \int_M f(x) d\rho_{\tau-t}(\gamma(t), dx) \\
 & = H_{\tau-t} f(\gamma(t)) .
 \end{aligned}$$

The case $t > \tau$ is trivial.

Combining this with (i) we get that F_t is a martingale.



Idea: I would like to define $F_t[\gamma] = \gamma(\tau)$ $\forall t$, but this is not Σ_t -meas. for $t < \tau$. Hence, for such t , I take an "average" of where I could be at τ , knowing where I am at t , which is I take the induced martingale. Since our measure is Γ_x Wiener, the heat flow H_t pops out.