Let's get $\mathbb{R}$eal
Exercise session part 1

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June 2023

1. Let $(R, \leq)$ be an ordered field. Then, the following properties hold:
   (i) $-1 < 0$;
   (ii) for $x \in R$ exactly one of the following holds: $x < 0$, $x = 0$, $x > 0$;
   (iii) R is a field of characteristic zero.
2. Prove (using only the definition) that $\mathbb{Q}$ and $\mathbb{R}$ with the usual ordering are ordered fields. Prove (using only the definition) that $\mathbb{C}$ cannot be ordered.
3. Prove that the ring $\mathbb{R}[x]$ admits (many? How many?) orderings. [Hint: choose where to put x. For instance, it might be > 0, but smaller then any other positive element...]
4. Ler $(R, \leq)$ be an ordered field and $P = \{x \in R \,|\, x \geq 0\}$. Prove that P is a proper cone satisfying

   $$P \cup (-P) = R.$$

   Conversely, given a proper cone P of R satisfying $P \cup (-P) = R$, prove that there exists an ordering of R for which P is the positive cone. Therefore, there is a one-to-one correspondence between orderings of a field and its proper cones.
5. Prove that a real closed field has a unique ordering.
6. Prove that every real closed field contains $\mathbb{Z}$ and $\mathbb{Q}$.
7. Explore Chapter 3.1 of the book *Real Algebraic Geometry* by Bochnak, Coste, and Roy for examples of real algebraic varieties with interesting features.
8. Prove that the ideal $I = \langle x^2 + y^2 \rangle \subset \mathbb{R}[x, y]$ is not a real ideal.
9. Let $I = \langle x^2 - 2x - 2 \rangle \subset \mathbb{R}[x]$. Write a certificate, in the Real Nullstellensatz fashion, for $\mathcal{V}_{\mathbb{R}}(I) = \emptyset$.

10. Let $I = \langle 81x^4 + 108x^2y^2 - 32x^2 - 8xy^4 + 8xy^2 - 4x + 4y^8 - 8y^6 + 44y^4 - 28y^2 + 6\rangle$. Describe the real variety $\mathcal{V}_{\mathbb{R}}(I)$ using the Nullstellensatz: does it have real points? [Hint: Figure 1.]

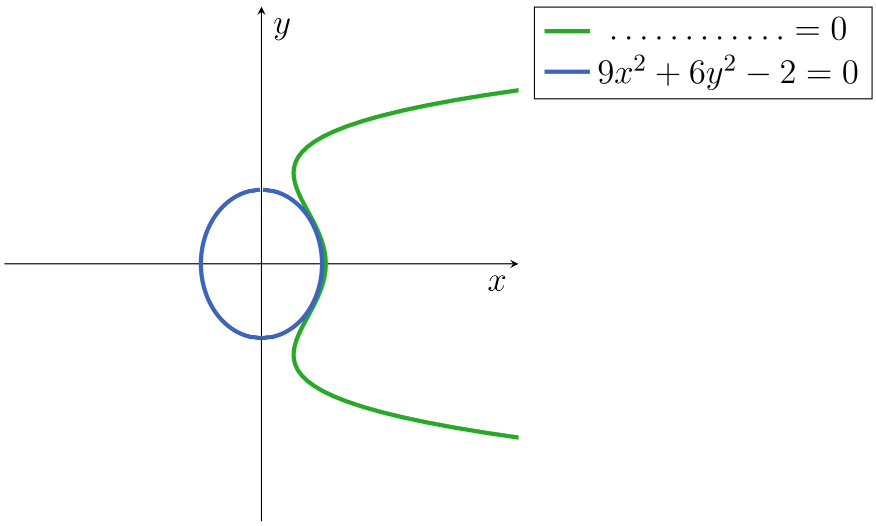

Figure 1: Two curves from the hint of Exercise 10.