Let's get $\mathbb{R}$eal
Exercise session part 2

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1. Prove that semialgebraic sets are closed under the following operations:
   (i) union, intersection, complement;
   (ii) interior, closure, boundary;
   (iii) Minkowski sum, product;
   (iv) taking connected components.
2. Prove that every semialgebraic set is the projection of an algebraic set.
3. Prove the following statements:
   (i) A semialgebraic subset of $\mathbb{R}$ is a finite union of points and intervals;
   (ii) The set $S = \{(x, y) \in \mathbb{R}^2 \mid y = \cos(x)\}$ is not semialgebraic.

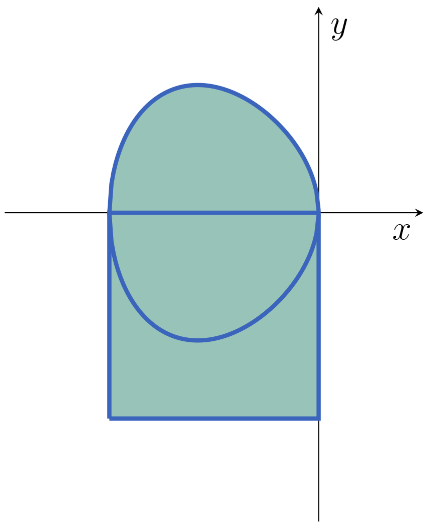

Figure 1: The set S_1 from Exercises 4 and 7.

4. Prove that the sets

$$S_1 = \{(x,y) \in \mathbb{R}^2 \mid y \leq x(x-1)(x+1), x \leq 0 \\ \text{or } -1 \leq x \leq 0, -1 \leq y \leq 0\},$$
$$S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x \leq 0 \text{ or } y \leq 0 \text{ or } z \leq 0\},$$

are not basic semialgebraic. See Figure 1.

5. Explore some software! Check out `QEPCAD`. Can we do quantifier elimination in `Mathematica`? How?

6. Consider the problem of the "complex product" of a segment and a square (example due to George E. Collins). Consider in $\mathbb{R}^2 \simeq \mathbb{C}$ the segment $C_1 = [0,2] \times \{i\}$ and the square $C_2 = [2,4] \times [-i,i]$. Write a formula and then compute (Exercise 5 might help!) a quantifier-free one for $C = \{a \cdot b \mid a \in C_1, b \in C_2\}$, where $\cdot$ denotes the product between complex numbers.

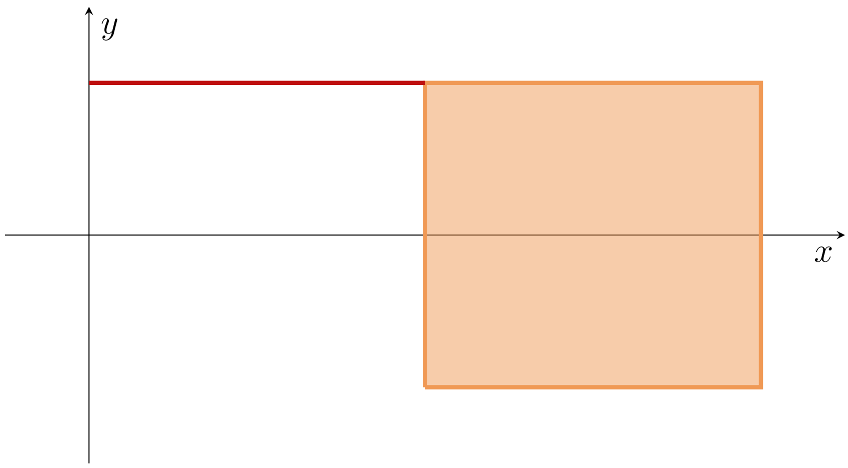

Figure 2: The sets C_1 (red) and C_2 (orange) from Exercise 6.

7. Write the cylindrical decomposition of the semialgebraic set S_1 from Exercise 4, Figure 1, with respect to both x and y.

8. Explore some software, part 2! Try to compute the cylindrical algebraic decomposition of some semialgebraic set. Start with a few polynomials of low degree in few variables. Increase the numbers and try to get a sense of how far one can push the computation.