Let's get $\mathbb{R}$eal
Exercise session part 3

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1. Prove that a univariate polynomial $f \in \mathbb{R}[x]$ of even degree is nonnegative on $\mathbb{R}$ if and only if it is a sum of squares.

2. Prove Pólya–Szegö Theorem: A univariate polynomial $f \in \mathbb{R}[x]$ is nonnegative on $[0, \infty)$ if and only if it can be written as

$$f(x) = \sigma_1(x) + x\,\sigma_2(x),$$

where σ_i is a sum of squares and $\deg \sigma_1, \deg(x\,\sigma_2) \leq \deg f$. [Hint: start with irreducible polynomials of low degrees, and then generalize.]

   (i) Apply Pólya–Szegö Theorem to prove that the polynomial

$$f(x) = 4x^7 + 22x^6 + 6x^5 - 17x^4 + 19x^3 + 2x^2 - 4x + 3$$

   is nonnegative on $[0, \infty)$. [Hint: some software might help...]

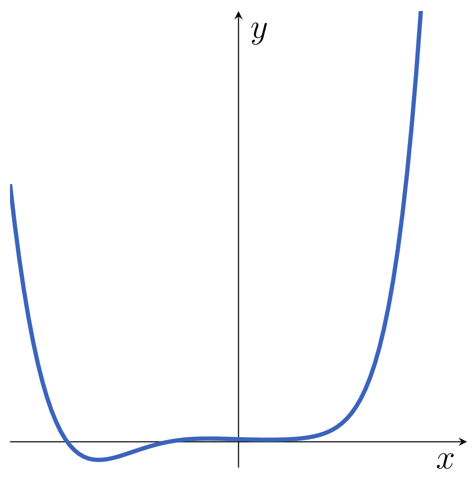

Figure 1: The graph of $f(x)$ from Exercise 2(i).

3. The Motzkin polynomial, step by step. Prove the following statements, for $f(x,y,z) = z^6 - 3x^2y^2z^2 + x^2y^4 + x^4y^2$:
   (i) Using the arithmetic-geometric mean inequality $\frac{a+b+c}{3} - \sqrt{abc} \geq 0$, prove that $f(x,y,x) \geq 0$.
   (ii) If $f = \sum f_i^2$, then no f_i could contain certain monomials. Make this precise, and find a contradiction. [For a more geometric/polyhedral proof, check out Theorem 7.2.1 of Thorsten Theobald's draft[1] on *Real Algebraic Geometry and Optimization*.]
4. Prove that the Motzkin polynomial from Exercise 3 can be written as a sum of squares of rational functions. [Hint: $(x^2+y^2)^2 \cdot f$ looks like a sum of squares. Can a software help? `Mathematica`? `Julia`? `Macaulay2`?]
5. Consider the polynomial

$$f(x,y) = x^8y^2+3x^6y^4-3x^6y^2+3x^4y^6-6x^4y^4+x^4+x^2y^8-3x^2y^6+2x^2y^2+y^4.$$

   Write the associated matrix A such that $f = m^T A m$, where m is the vector of monomials, up to degree $\deg f$.
   (i) Is A positive semidefinite?
   (ii) Find a sum of squares decomposition of f.
6. Go back to Exercise 10 of the first exercise sheet, and denote by f the generator of the ideal I. Is $f - 1$ a sum of squares?
7. Prove that the following two definitions/interpretations of a spectrahedron S are equivalent:
   (i) S is the intersection of the cone of positive semidefinite martices with a linear (affine) space;
   (ii) There exist some symmetric matrices $A_0, \ldots, A_n$ such that

$$S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i A_i + A_0 \succcurlyeq 0\}.$$

8. Properties of spectrahedra. Prove that
   (i) a polytope is a spectrahedron;
   (ii) a spectrahedron is a basic closed semialgebraic convex set;
   (iii) the cylinder $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1, -1 \leq z \leq 1\}$ is a spectrahedron;
   (iv) the projection of a spectrahedron is not necessarily a spectrahedron. [Hint: use the previous points.]

[1] https://www.math.uni-frankfurt.de/~theobald/ragopt/main.pdf