## Exercises

## M-convex functions, Lorentzian polynomials, and valuated matroids

The exercises below are not ordered (except by roughly being grouped according to topic). Please feel free to work on whatever interests you!

## 1 M-convexity

Exercise 1.1. Show that a nonempty set $J \subseteq \mathbb{N}^{n}$ is an M-convex set if and only if it satisfies the following exchange axiom:

$$
\begin{gathered}
\forall x, y \in J, u \in \operatorname{supp}^{+}(x-y), \exists v \in \operatorname{supp}^{-}(x-y) \text { such that } \\
x-\chi_{u}+\chi_{v} \in J \text { and } y+\chi_{u}-\chi_{v} \in J,
\end{gathered}
$$

where $\operatorname{supp}^{+}(x):=\left\{i \in[n] \mid x_{i}>0\right\}$ is the positive support of a vector, $\operatorname{supp}^{-}(x):=\{i \in$ $\left.[n] \mid x_{i}<0\right\}$ is the negative support of a vector, and $\chi_{i}$ is the characteristic vector of $i \in[n]$, e.g. $\chi_{1}=(1,0, \ldots, 0)$.

Exercise 1.2. Think about why M-convex sets are more general than matroids.

## Exercise 1.3.

- Characterize all M-convex subsets of $\mathbb{N}^{2}$. Now characterize all M-convex subsets of $\mathbb{N}^{3}$. Can you observe anything about the "shape" of the convex hulls of M-convex sets in $\mathbb{N}^{n}$ ?
- A generalized permutohedron is a deformation of a usual permutohedron; i.e., it is a polytope obtained by moving vertices of a usual permutohedron so that directions of all edges are preserved (and we allow some of the edges to possibly degenerate to a point). See Section 6 of Postnikov's paper here: https://arxiv.org/pdf/math/0507163.pdf for more about permutohedra and generalized permutohedra.
Try to convince yourself of the validity of the following statement: the convex hull of any M-convex set is an integral generalized permutohedron, and conversely the set of integer points of any integral generalized permutohedron is an M-convex set.
- Give an example of an M-convex set whose convex hull is an integral generalized permutohedron, but not an integral permutohedron.
Exercise 1.4. Characterize all homogeneous polynomials $f(x, y) \in \mathbb{R}_{\geq 0}[x, y]$ that have Mconvex support.


## Exercise 1.5.

a) Prove that an M-convex function $f: \mathbb{N}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ with $\operatorname{dom}(f) \neq \emptyset$ satisfies

$$
\begin{align*}
& \forall x, y \in \operatorname{dom}(f) \text { and } u \in \operatorname{supp}^{+}(x-y) \exists v \in \operatorname{supp}^{-}(y-x) \text { such that } \\
& f(x)+f(y) \geq f\left(x-\chi_{u}+\chi_{v}\right)+f\left(y+\chi_{u}-\chi_{v}\right), \tag{1}
\end{align*}
$$

where $\operatorname{supp}^{+}(x):=\left\{i \in[n] \mid x_{i}>0\right\}$ is the positive support of a vector, $\operatorname{supp}^{-}(x):=$ $\left\{i \in[n] \mid x_{i}<0\right\}$ is the negative support of a vector, and $\chi_{i}$ is the characteristic vector of $i \in[n]$, e.g. $\chi_{1}=(1,0, \ldots, 0)$.
b) (1) is called the exchange axiom. Show that any function $\mathbb{N}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ satisfying the exchange axiom is M-convex. In other words, the symmetric exchange axiom and the exchange axiom are equivalent.


Illustration of the exchange property from Murota's book "Discrete Convex Analysis".
c) Show that the (symmetric) exchange axiom is equivalent to the local exchange property: $\forall x, y \in \operatorname{dom}(f)$ with $\|x-y\|_{1}=4, \exists u \in \operatorname{supp}^{+}(x-y)$ and $v \in \operatorname{supp}^{-}(x-y)$ such that $f(x)+f(y) \geq f\left(x-\chi_{u}+\chi_{v}\right)+f\left(y+\chi_{u}-\chi_{v}\right)$.

Exercise 1.6. The indicator function $\nu_{J}: \mathbb{N}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ of a subset $J \subseteq \mathbb{N}^{n}$ is defined by

$$
\nu_{J}(\alpha)=\left\{\begin{array}{lc}
0 & \text { if } \alpha \in J \\
\infty & \text { otherwise }
\end{array}\right.
$$

Show that $J$ is an M-convex set if and only if $\nu_{J}$ is an M-convex function.
Exercise 1.7. Show that the effective domain of an M-convex function on $\mathbb{N}^{n}$ is contained in $\Delta_{n}^{d}$ for some $d$.
Exercise 1.8. A function $f: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ is convex if $\lambda f(x)+(1-\lambda) f(y) \geq f(\lambda x+(1-$ ג) $y) \forall x, y \in \mathbb{R}^{n}, \lambda \in[0,1]$.
Show that the effective domain of a convex function is a convex set and that an M-convex function can be extended to a convex function.

## 2 Matroids and Valuated matroids

Exercise 2.1. Recall the graphic matroid given by graph below, or equivalently the vector matroid given by the set of vectors. We write $M=(E, \mathcal{B})$ for this matroid.


In the lecture, we computed the bases and independent sets of this matroid.

- A subset $D \subseteq E$ that is not independent is called dependent. Compute the dependent sets of $M$.
- A dependent set $C \subset E$ that is minimal with respect to inclusion is called a circuit. Identify the circuits of $M$.
- A flat is a set $F \subset E$ such that $|C \backslash F| \neq 1$ for any circuit $C$. Compute the flats of $M$.

Exercise 2.2. Compute the dependent sets, circuits, and flats of the uniform matroid $U_{r, n}$. (See Exercise 2.1 above for the definition of dependent set, circuit, and flat.)

Exercise 2.3. Show that all bases of a matroid have the same cardinality.
Exercise 2.4. Show that $U_{2,4}$ is not a graphic matroid. Can you find all uniform matroids that are not graphic?

Exercise 2.5. Consider the $n$-th braid matroid $K_{n}$. This is the graphic matroid associated to the complete graph on $n$ vertices. Give a precise description of the lattice of flats of $K_{n}$.

Exercise 2.6. Show that every graphic matroid is isomorphic to a vector matroid.
Exercise 2.7. Prove that the underlying matroid of a valuated matroid is indeed a matroid.
Exercise 2.8. Let $J \subseteq\{0,1\}^{n}$. Show that $J$ is the set of bases of a matroid on $[n]$ if and only if $J$ is an M-convex set.

Exercise 2.9. Compute and describe the $\operatorname{Dressian} \operatorname{Dr}(M)$ for the matroids $M_{1}=U_{2,4}$, and $M_{2}=U_{2,5}$.
(You can use either definition of the Dressian: the one from the lecture as parameter space of valuated matroids or the one on in Definition 4.7, see Exercise 4.8.)

## 3 Log-concavity

Exercise 3.1. Consider the binomial coefficients

$$
\binom{n}{k}=\frac{n!}{k!(n-k)!} .
$$

Show that the sequence

$$
\binom{n}{0},\binom{n}{1}, \ldots,\binom{n}{n-1},\binom{n}{n}
$$

(the $n$-th row of Pascal's triangle) is log-concave.
In fact, in your proof, you probably showed that the above sequence is actually ultra log-concave: A sequence $\left(a_{0}, a_{1}, \ldots, a_{n}\right)$ of nonnegative real numbers is ultra log-concave if for all $0<i<n$, we have

$$
\frac{a_{i}^{2}}{\binom{n}{i}^{2}} \geq \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} .
$$

## Exercise 3.2.

- Show that if $f=\sum_{i=0}^{d} c_{i} x^{i} \in \mathbb{R}_{\geq 0}[x]$ is a real-rooted polynomial (it has only real roots), then its sequence of coefficients $\left(c_{0}, c_{1}, \ldots, c_{d}\right)$ is ultra log-concave. (See Exercise 3.1 for the definition of ultra log-concavity.) Hint: Use Rolle's theorem or the Mean Value theorem from calculus.
- Redo Exercise 3.1 above in a different way using the previous part of this problem.

Exercise 3.3. A sequence $\left(a_{0}, a_{1}, \ldots, a_{n}\right)$ of nonnegative real numbers has no internal zeros if $a_{k_{1}} a_{k_{3}} \neq 0$ implies that $a_{k_{2}} \neq 0$ for all $0 \leq k_{1}<k_{2}<k_{3} \leq n$.. It is unimodal if there exists $0 \leq k \leq n$ such that $a_{0} \leq a_{1} \leq \cdots \leq a_{k-1} \leq a_{k} \geq a_{k+1} \geq \cdots \geq a_{n}$.

Show that if a sequence ( $a_{0}, a_{1}, \ldots, a_{n}$ ) of nonnegative real numbers has no internal zeros and is log-concave, then it is unimodal.

Exercise 3.4. Let $f=\sum_{i=0}^{d} a_{i} x^{i} \in \mathbb{R}_{\geq 0}[x]$ and $g=\sum_{i=0}^{e} b_{i} x^{i} \in \mathbb{R}_{\geq 0}[x]$ be two polynomials such that both $\left(a_{0}, a_{1}, \ldots, a_{d}\right)$ and ( $b_{0}, b_{1}, \ldots, b_{e}$ ) are log-concave with no internal zeroes. Prove that the coefficient sequence of the product $f g$ is also log-concave with no internal zeros.

## 4 Tropicalization

Exercise 4.1. Let $p \in \mathbb{N}$ be a prime. The $p$-adic valuation on $\mathbb{Q}$ is a map given as follows: Zero maps to $\infty$ and for $q \in \mathbb{Q}^{*}$, the valuation is $\operatorname{val}_{p}(q)=k$ where $k \in \mathbb{Z}$ such that $q=p^{k} \frac{a}{b}$ where $\operatorname{gcd}(a, b)=1$ and $p \nmid a, p \nmid b$.
Show that the $p$-adic valuation is a valuation on $\mathbb{Q}$.
Exercise 4.2. Let $K$ be a field with valuation val. Let $a, b \in K$. Prove: If $\operatorname{val}(a) \neq \operatorname{val}(b)$, then $\operatorname{val}(a+b)=\min (\operatorname{val}(a), \operatorname{val}(b))$.

Exercise 4.3. Tropicalize $f=x_{1} x_{3}+t^{-1} x_{1} x_{4}+t^{12} x_{1} x_{5}+t^{-2} x_{1} x_{6}+t^{-3} x_{2} x_{3}+t^{-4} x_{2} x_{4}+$ $t^{9} x_{2} x_{5}+t^{-5} x_{2} x_{6}+t^{-5} x_{3} x_{5}+x_{3} x_{6}+t^{-6} x_{4} x_{5}+t^{-1} x_{4} x_{6}+t^{-7} x_{5} x_{6}$ and show that it gives an M-convex function. Does $-\operatorname{trop}(f)$ give a valuated matroid?

## Some brief background to tropical geometry

For further reading:

- "Introduction to tropical geometry" by Maclagan and Sturmfels
- "Essentials of tropical combinatorics" by Joswig
- several surveys and summer school lecture notes on arXiv...

Definition 4.4. Let $f_{t}=\sum_{\alpha \in \Delta_{n}^{d}} s_{\alpha}(t) x^{\alpha}$ be a nonzero homogeneous polynomial with coefficients in $\overline{\mathbb{K}}$.
The tropical hypersurface of $f_{t}$ is given as follows
$V\left(\operatorname{trop}\left(f_{t}\right)\right)=\operatorname{closure}\left\{w \in \mathbb{R}^{n} \mid\right.$ the max. in $\max _{\alpha \in \Delta_{n}^{d}}\left\{-\operatorname{val}\left(s_{\alpha}(t)\right)+\alpha \cdot w\right\}$ is achieved at least twice $\}$.
Since $f_{t}$ is homogeneous, $w \in V\left(\operatorname{trop}\left(f_{t}\right)\right) \Leftrightarrow w+\lambda \mathbf{1} \in V\left(\operatorname{trop}\left(f_{t}\right)\right)$. Therefore, we consider $V\left(\operatorname{trop}\left(f_{t}\right)\right) \subset \mathbb{R}^{n} / \mathbf{1} \mathbb{R} .\left(\mathbb{R}^{n} / \mathbf{1} \mathbb{R}\right.$ is called the tropical projective torus.)

Lemma 4.5. Let $f=\sum_{\alpha \in \Delta_{n}^{d}} s_{\alpha}(t) x^{\alpha} \in \overline{\mathbb{K}}\left[x_{1}, \ldots, x_{n}\right]$ be a homogeneous polynomial and let $\omega \in \overline{\mathbb{K}}$ such that $f(\omega)=0$. Let $u=\left(-\operatorname{val}\left(\omega_{1}\right), \ldots,-\operatorname{val}\left(\omega_{n}\right)\right)$.
Then the maximum in $\max _{\alpha \in \Delta_{n}^{d} \text { with } s_{\alpha}(t) \neq 0}\left\{-\operatorname{val}\left(s_{\alpha}(t)\right)+\alpha \cdot u\right\}$ is achieved at least twice.
This shows that the choice of the word 'hypersurface' is not random but fits with the algebraic hypersurface of a polynomial.

Example 4.6. Let $f=x_{1}+x_{2}+x_{3}+x_{4}$. Then $f$ is linear, so the tropical hypersurface is called a tropical hyperplane. It is a polyhedral complex in $\mathbb{R}^{4} / \mathbf{R} \cong \mathbb{R}^{3}$ consisting of one vertex, four rays and six 2-dimensional cells.


Let $f=3 x_{1}^{2}+7 t x_{1} x_{2}-x_{2}^{2}+t x_{1} x_{3}+\left(t^{4}-7 t^{6}\right) x_{2} x_{3}+\left(-3 t^{5}+t^{6}-\frac{1}{5} t^{11}\right) x_{3}^{2}$. Then the tropical hypersurface of $f$ is a tropical quadratic curve.


Definition 4.7. Let $M$ be a matroid on $[n]$, with bases $\mathcal{B}$. The Dressian of $M$, denoted $\operatorname{Dr}(M)$, is obtained by intersecting the tropical hypersurfaces of the three-term Plücker relations in $\mathbb{R}^{|\mathcal{B}|} / \mathbf{1} \mathbb{R}$.

The three-term Plücker relations associated to the $\operatorname{Dressian~} \operatorname{Dr}(U d, n)$ are

$$
0=p_{S a b} \cdot p_{S c d}-p_{S a c} \cdot p_{S b d}+p_{S a d} \cdot p_{S b c},
$$

where $1 \leq a<b<c<d \leq n$ and $S \in\binom{[n]}{d-2}$ disjoint to $\{a, b, c, d\}$.
Exercise 4.8. Show that the two definitions of the Dressian (via valuated matroid and via 3-term Plücker relations) really define the same object. For this start with the uniform matroid $U_{d, n}$ and show

$$
\bigcap_{P \text { three-term Plücker relations }} V(\operatorname{trop}(P))=\left\{\text { valuated matroids over } U_{d, n}\right\}
$$

What would change if we now took an arbitrary rank $d$ matroid on $n$ elements as the underlying matroid?

Exercise 4.9. Prove Lemma 4.5. the following:
Let $f=\sum_{\alpha \in \Delta_{n}^{d}} s_{\alpha}(t) x^{\alpha} \in \overline{\mathbb{K}}\left[x_{1}, \ldots, x_{n}\right]$ be a homogeneous polynomial and let $\omega \in \overline{\mathbb{K}}$ such that $f(\omega)=0$. Let $u=\left(-\operatorname{val}\left(\omega_{1}\right), \ldots,-\operatorname{val}\left(\omega_{n}\right)\right)$. Then the maximum in $\max _{\alpha \in \Delta_{n}^{d} \text { with } s_{\alpha}(t) \neq 0}\left\{-\operatorname{val}\left(s_{\alpha}(t)\right)+\alpha \cdot u\right\}$ is achieved at least twice.

Exercise 4.10. Convince your neighbor that for a matroid $M=(E, \mathcal{B})$ we have
$\operatorname{Dr}(M)=\operatorname{closure}\left\{-\operatorname{trop}\left(f_{t}\right) \mid f_{t}\right.$ is a Lorentzian polynomial with $\left.\operatorname{supp}\left(f_{t}\right)=\mathcal{B}\right\}$.
Exercise 4.11. Go to Section 3.3 in the article "Lorentzian polynomials" (https://arxiv. org/abs/1902.03719) and work towards understanding the role of the Dressian and phylogenetic trees in the proofs of Theorem 3.14 and 3.20.

## 5 Lorentzian polynomials

Exercise 5.1. Show that the following polynomials are Lorentzian.

- $f=\frac{x_{1}^{2}}{2}+x_{1} x_{2}+\frac{x_{2}^{2}}{2}$,
- $g=\frac{x_{1}^{2} x_{3}}{2}+\frac{x_{1}^{2} x_{2}}{2}+\frac{x_{1} x_{2}^{2}}{2}+\frac{x_{2}^{2} x_{3}}{2}+\frac{x_{1} x_{3}^{2}}{2}+\frac{x_{2} x_{3}^{2}}{2}+2 x_{1} x_{2} x_{3}$, and
- $h=x_{1} x_{3}+x_{1} x_{2}+\frac{x_{2}^{2}}{2}+\frac{x_{3}^{2}}{2}+2 x_{2} x_{3}$.

Exercise 5.2. If you know about symmetric polynomials, try to prove that elementary symmetric polynomials are Lorentzian.

Exercise 5.3. Give an example of a Lorentzian polynomial whose support has convex hull that is an integral generalized permutohedron, but not an integral permutohedron. (See Exercise 1.3 for more about generalized permutohedra, including a reference for the definition.)

Exercise 5.4. Give an example of a polynomial with M-convex support that is not Lorentzian.

Exercise 5.5. Characterize all polynomials $f(x, y) \in \mathbb{R}_{\geq 0}[x, y]$ that are Lorentzian.
Exercise 5.6. Show that a bivariate polynomial $\sum_{k=0}^{d} c_{k} x_{1}^{k} x_{2}^{d-k} \in \mathbb{R}_{\geq 0}\left[x_{1}, x_{2}\right]$ is Lorentzian if and only if the sequence of coefficients $\left(c_{0}, c_{1}, \ldots, c_{d}\right)$ has no internal zeros and is ultra log-concave.

## Exercise 5.7.

- Prove that if a polynomial $f=\sum_{\alpha \in \mathbb{N}^{n}} \frac{c_{\alpha}}{\alpha!} x^{\alpha}$ is Lorentzian, then we have the discrete log-concavity:

$$
c_{\alpha}^{2} \geq c_{\alpha+e_{i}-e_{j}} c_{\alpha-e_{i}+e_{j}} \quad \text { for all } \alpha \text { and for all } i, j \in[n] .
$$

- Revisit this problem in the following way: what inequalities can you conclude if $f=$ $\sum_{\alpha \in \mathbb{N}^{n}} c_{\alpha} x^{\alpha}$ is Lorentzian (i.e., without the normalization)?

Exercise 5.8. Show that the following polynomials over $\mathbb{K}$ are Lorentzian.

$$
\begin{gathered}
f_{t}=\frac{t^{-1}}{2} y z^{2}+\frac{t^{-1}}{2} y^{2} z+\frac{t^{-1}}{2} x z^{2}+\frac{t^{-1}}{2} x^{2} z+\frac{t^{-1}}{2} x^{2} y+\frac{t^{-1}}{2} x y^{2}+\frac{t^{-2}}{2} x y z \\
g_{t}=9 t^{-1} y z^{2}+9 t^{-1} y^{2} z+9 t^{-1} x z^{2}+9 t^{-1} x^{2} z+9 t^{-1} x^{2} y+9 t^{-1} x y^{2}+27 t^{-2} x y z
\end{gathered}
$$

## 6 Bergman fan

For further reading:

- "Introduction to tropical geometry" by Maclagan and Sturmfels
- "Essentials of tropical combinatorics" by Joswig
- "Lorentzian fans" (Dustin Ross) https://arxiv.org/pdf/2304.13176.pdf
- "Hodge theory for combinatorial geometries" (Adiprasito, Huh, Katz) https://arxiv. org/pdf/1511.02888.pdf

Definition 6.1. Let $M=(E, \mathcal{B})$ be a matroid. A subset $D \subset E$ that is not independent, is called dependent.
A dependent set $C \subset E$ that is minimal with respect to inclusion is called a circuit.
A flat of a matroid $M$ is a set $F \subset E$ such that $|C \backslash F| \neq 1$ for any circuit $C$.
Example 6.2. Consider the matroid $M=(E, \mathcal{B})$ from the beginning of Jacob's lecture:


$$
\begin{gathered}
v_{1} \\
v_{2}
\end{gathered} v_{3} v_{4} v_{5}, ~\left[\begin{array}{ccccc}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{array}\right]
$$

Its circuits are $\{1,2,3\},\{3,4,5\},\{1,2,4,5\}$.
Do you see how you can get the circuits from the graph?
Its flats are

- $\emptyset($ as no circuit has size 1$)$
- $\{i\} \forall i \in[5]$ (as no circuit has size 2 )
- $\{\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{1,2,3\},\{3,4,5\}\}$
- [5]

Note: A circuit can be a flat!


The flats can be ordered by inclusion. This is called the lattice of flats.
Definition 6.3. Associated to the poset of flats of a matroid of rank $d$ on $n$ elements is a simplicial complex: the order complex. Its vertices are the elements of the poset, its simplices are the proper chains, i.e. the totally ordered subsets o fht eposet without using $\emptyset$ and $[n]$. The order complex of the lattice of flats is pure (i.e. all maximal cells of the complex have the same dimension) of dimension $d-1$.

In the example, the order complex is of dimension 1 , vertices are for example $\{1\}$ and $\{1,4\}$, and an edge would be $\{\{1\},\{1,4\}\}$.

Definition 6.4. A flat $F$ of the matroid $M$ is represented by its incidence vector $e_{F}=$ $\sum_{i \in F} e_{i}$, where we regard $e_{F}, e_{i}$ as elements in $\mathbb{R}^{n} / \mathbf{1} \mathbb{R}$. For any chain of flats $\emptyset \subsetneq F_{1} \subsetneq \cdots \subsetneq$ $F_{d} \subsetneq E$, we consider the polyhedral cone spanned by their incidence vectors

$$
\sigma=\operatorname{cone}\left(e_{F_{1}}, \ldots, e_{F_{d}}\right)+\mathbf{1} \mathbb{R}=\left\{\lambda_{0} \mathbf{1}+\lambda_{1} e_{F_{1}}+\ldots+\lambda_{r} e_{F_{d}}: \lambda_{1}, \ldots, \lambda_{d} \geq 0\right\}
$$

Since $1, e_{F_{1}}, \ldots, e_{F_{d}}$ are linearly independent, this is an $r$-dimensional simplicial cone (i.e. the cone over an $(d-1)$-dimensional simplex).

The collection of all such cones from chains of flats of $M$ forms a pure simplicial fan of dimension $\operatorname{rk}(M)-1$ in $\mathbb{R}^{n} / \mathbf{1} \mathbb{R}$.

This fan will be called Bergman fan in this lecture series.
(Careful: The fan structure introduced here on the fan is not the standard Bergman fan structure you'll find in the literature, but a finer structure.)

The support of the Bergman fan equals the tropical linear space $\operatorname{trop}(M)$.
Example 6.5. For the matroid in the example above, the Bergman fan is of dimension 2 in $\mathbb{R}^{4} \cong \mathbb{R}^{5} / \mathbf{1}$. It has 11 rays and 14 two-dimensional cells.

Exercise 6.6. Check that the Bergman fan of $U_{3,4}$ coincides with the tropical hyperplane $V\left(\operatorname{trop}\left(x_{1}+x_{2}+x_{3}+x_{4}\right)\right)$.


This is not an accident! There is a way to associate a matroid to a linear subspace in $\left(\mathbb{K}^{*}\right)^{n}$, such that the support of the Bergman fan of this matroid coincides with the tropicalization of the linear space. For more see Chapter 4 in "Introduction to tropical geometry".

Exercise 6.7. Let $M$ be the matroid realized by the matrix

$$
\left(\begin{array}{ccccc}
0 & -2 & 1 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 \\
0 & -1 & -1 & 1 & 1
\end{array}\right)
$$

Compute the circuits and the lattice of flats. Compute the Bergman fan of the matroid.
Exercise 6.8. Draw the Bergman fan for $U_{2,4}$.

