Chow Lectures 2023

Preparatory Lectures - Sofía Garzón Mora

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1 Exercises: Volume forms in convex geometry

1.1 Lorentzian Polynomials

Exercise 1. Construct as many Lorentzian polynomials in three variables as you can.

Exercise 2. The Lorentzian polynomials in 3 variables of degree 3 with support in the monomials $x_1x_2x_3$ and $x_i^2x_j$ for $1 \leq i, j \leq 3$ form a convex set in the seven dimensional vector space of such polynomials. Describe this set.

The goal of the following exercises is now to prove the following theorem:

Theorem 1.1 (Brändén, Huh)**.** *The volume polynomial* vol_K *is a Lorentzian polynomial for any collection of convex bodies* $K = (K_1, ..., K_n)$ *in* $\mathbb{R}^d$.

For that we need the following results:

Proposition 1.2. *The following are equivalent for any* $w \in \mathbb{R}^n$ *satisfying* $f(w) > 0$*:*

1. *The Hessian of* $f^{1/d}$ *is negative semidefinite at* w.
2. *The Hessian of* $\log f$ *is negative semidefinite at* w.
3. *The Hessian of* f *has exactly one positive eigenvalue at* w.

And:

Theorem 1.3. *(Special case of Brunn-Minkowski Theorem) Let* $K = (K_1, ..., K_n)$ *be a collection of convex bodies. For any convex bodies* $C_3, ..., C_d$ *in* $\mathbb{R}^d$*, the function*

$$w \mapsto \mathrm{V}\left(\sum_{i=1}^n w_iK_i, \sum_{i=1}^n w_iK_i, C_3, ..., C_d\right)^{1/2}$$

is concave on $\mathbb{R}^n_{>0}$.

Exercise 3. For any convex bodies $C_0, C_1, ..., C_d$ in $\mathbb{R}^d$, the mixed volume $V(C_1, C_2, ..., C_d)$ is symmetric in its arguments. Prove that it satisfies the relation

$$V(C_0 + C_1, C_2, ..., C_d) = V(C_0, C_2, ..., C_d) + V(C_1, C_2, ..., C_d).$$

Exercise 4. Compute the polynomial $\mathrm{vol}(\lambda A + \mu B)$, where $\lambda, \mu \in \mathbb{R}_{>0}$, A is an $a \times c$ rectangle and B is a $b \times d$ rectangle, and deduce the mixed volumes explicitly.
Do the same for your favorite pair (triple) of polygons.

Exercise 5. Show that for $\alpha \in \Delta_n^{d-2}$,

$$\left(\frac{2!}{d!}\partial^\alpha \mathrm{vol}_K(w)\right)^{1/2} = \mathrm{V}\left(\sum_{i=1}^n w_iK_i, \sum_{i=1}^n w_iK_i, \underbrace{K_1, ..., K_1}_{\alpha_1}, ..., \underbrace{K_n, ..., K_n}_{\alpha_n}\right)^{1/2}.$$

Exercise 6. Prove Theorem 1.1. (Hint: Assume by continuity of the volume functional, that every convex body in K is d-dimensional. We only need to show that $\partial^\alpha \mathrm{vol}_K$ is Lorentzian for every $\alpha \in \Delta_n^{d-2}$.)

Exercise 7. (AF Inequality) Prove that if $f = \sum_{\alpha \in \Delta_n^d} \frac{c_\alpha}{\alpha!} w^\alpha$ is a Lorentzian polynomial, then

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j}c_{\alpha-e_i+e_j}$$

for any $i, j = 1, ..., n$, $\alpha \in \Delta_n^d$.

1.2 Lorentzian Polynomials on cones

Exercise 8. Compute the lineality space L_f and the associated simplicial complex Δ_f for $f = 4w_1^2 + 4w_1w_2 + w_2^2$. Is this pair (Δ_f, L_f) hereditary? Understand the definition of the associated cone $\mathcal{K}_f$ and, optionally, try to compute it for this example.

Exercise 9. Let $\mathcal{K}_P$ be the set of all simple polytopes that have the same normal fan as P. Prove that $\mathcal{K}_P$ is an open convex cone in $\mathbb{R}^n$.

Now, the goal of the following exercises is to prove a second main result:

Theorem 1.4 (Brändén, Leake)**.** *(Alexandrov-Fenchel Inequalities) Let $K_1, K_2, ..., K_n$ be convex bodies in $\mathbb{R}^n$. Then,*

$$V(K_1, K_2, ..., K_n)^2 \geq V(K_1, K_1, K_3, ..., K_n)V(K_2, K_2, K_3, ..., K_n).$$

For this, we need the following:

Lemma 1.5. *Let $\mathcal{K} \neq \emptyset$ be an open convex cone in $\mathbb{R}^n$, let $(x, y) \mapsto \mathrm{P}(x, y)$ a symmetric bilinear form on $\mathbb{R}^n$ such that $\mathrm{P}(v, v) > 0$ for all $v \in \mathcal{K}$. The following are equivalent:*

(AF) $\mathrm{P}(v, w)^2 \geq \mathrm{P}(v, v)\,\mathrm{P}(w, w)$ *for all* $v, w \in \mathcal{K}$.

(AF2) $\mathrm{P}(v, x)^2 \geq P(v, v)P(x, x)$ *for all* $v \in \mathcal{K}, x \in \mathbb{R}^n$.

(H) P *has exactly one positive eigenvalue.*

Exercise 10. Argue that we may replace the condition (HR) in the definition of a $\mathcal{K}$-Lorentzian polynomial by

(AF) For all $v_1, ..., v_d \in \mathcal{K}$,

$$(\partial_{v_1}\partial_{v_2}\partial_{v_3}\cdots\partial_{v_d} f)^2 \geq (\partial_{v_1}\partial_{v_1}\partial_{v_3}\cdots\partial_{v_d} f)(\partial_{v_2}\partial_{v_2}\partial_{v_3}\cdots\partial_{v_d} f)$$

Exercise 11. Give an example of an hereditary Lorentzian polynomial.

Exercise 12. Compute L_P for your favorite simple polytope P. (Or compute it for the cube if you really cannot decide amongst your favorites). Compute $\tau\Delta_P$ and the link $\mathrm{lk}_{\Delta_P}(v)$ where v is a vertex.

Exercise 13. If P is a simple polytope, prove that (Δ_P, L_P) is hereditary and if time allows, that $\mathcal{K}_P \subseteq \mathcal{K}_{\mathrm{vol}_P}$.

We also need the following strong result:

Theorem 1.6 (Brändén, Leake)**.** *If P is a simple polytope,* vol_P *is hereditary Lorentzian.*

An idea of the proof of this theorem is as follows: By Exercise 11, $\mathcal{K}_{\mathrm{vol}_P}$ is not empty. If we let P_i the facet of P with normal ρ_i, then $\partial_i \mathrm{vol}_P(w) = \mathrm{vol}_{P_i}(((w_j - w_i\cos(\theta_{ij}))/\sin(\theta_{ij}))_j)$ where θ_{ij} is the angle between ρ_i and ρ_j. Then we need to see that vol_P has at most one positive eigenvalue for $d = 2$. By performing suitable edge subdivisions, $\mathrm{vol}_P \sim f$ for some polynomial f such that $\Delta_f = \Delta_Q$, where Q is a triangle. Since vol_Q is hereditary Lorentzian, so is vol_P.

Exercise 14. Prove Theorem 1.4. (Hint: By an approximation argument, reduce to the case of simple polytopes. Use Theorem 1.6 and Exercise 10).