Consistent Variance Curve Models

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Abstract

We introduce equity forward variance term-structure models and derive the respective HJM-type arbitrage conditions. We then discuss finite-dimensional Markovian representations of the infinite-dimensional fixed time-to-maturity forward variance swap curve and analyse examples of such variance curve functionals.

The results are then applied to show that the speed of mean-reversion in standard stochastic volatility models must be kept constant when the model is recalibrated (a finding similar to Filipovic's [13] observation for interest-rate models). We also show that some standard implied volatility term-structure functionals can lead to arbitrage when refitted on a regular basis.

1 Introduction and Presentation of Results

Standard financial equity market models model exclusively the price process of the stock prices. The prices of liquid derivatives, like plain vanilla calls, are only used to calibrate the parameters of the model. A more natural approach

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would be to model the evolution of stock price and some liquid instruments simultaneously.

Various approaches, such as those by Cont [7], Härdle [16], or Haffner [17], take on the problem by modeling the stochastic evolution of implied volatility surfaces (and therefore European call prices) according to certain stylized facts or statistical observations. The price processes obtained from these models, however, are not guaranteed to remain martingales.

In this article, we consider *variance swaps* as liquid derivatives and derive conditions such that the joint market of stock price and variance swap prices is free of arbitrage. Such models can then be used to price exotic options and allow the computation of hedges with respect to stock and variance swaps in a consistent way.

Variance Swap Market Models

For the world's equity stock indices, a fairly liquid market of *variance swaps* has evolved in recent years. Given an index S, such a variance swap exchanges the payment of *realized variance* of the log-returns against a previously agreed strike price. The (zero mean) annualized realized variance for the period $[0, T]$ with business days $0 = t_0 < \ldots < t_n = T$ is usually defined as

$$\mathcal{V}^n(T) := \frac{d}{n} \sum_{i=1,\ldots,n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 ,$$

but contracts may vary. The constant d denotes the number of trading days per year. A standard result (eg Protter [24], p. 66) gives that

$$\langle \log S \rangle_T = \lim_{m \uparrow \infty} \sum_{t_i \in \tau_m} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2$$

where the limit is taken over refining subdivisions $\tau_m = (0 = t_0^m < \cdots < t_m^m = T)$, that is $\lim_{m \uparrow \infty} \sup_{i=1,\ldots,m} |t_i - t_{i-1}| = 0$. We can focus without loss of generality on variance swaps with a zero strike price.

We will assume in this article that the realized variance paid by a variance swap is the realized quadratic variation of the logarithm of the index price, i.e. $\mathcal{V}^n(T) \equiv \langle \log S_T \rangle$. We also assume that the stock price process is continuous.[1] Our starting point for this article will be that variance swaps are liquidly traded for all maturities.

Problem *Given today's variance swap prices $V_0(T)$ for all maturities $T \in [0, \infty)$, we want to model the price processes $V(T) = (V_t(T))_{t \in [0,\infty)}$ (with $V_t(T) = V_T(T)$ for $t > T$) such that the joint market with all variance swaps and the*

[1] If we also assume that we know all call prices for all strikes, Neuberger [22] has shown that the price of a variance swap can be computed from the market data. See also Demeterfi et al. [9] for a good overview on this approach. However, as discussed in Buehler [6], we usually only have a discrete number of traded options on the market.

index price itself is free of arbitrage.

Apart from the additional presence of the stock price, this resembles closely the situation in interest rate theory where the aim is to construct arbitrage-free price processes of zero bonds (see Heath et al. [18]). We carry this similarity further and introduce the *forward variance* of the log-returns of S as

$$v_t(T) := \partial_T V_t(T) \ .$$

We then have a HJM-type result, namely that (under the assumptions of the next section) v must be a martingale and therefore has no drift. This observation helps to solve the above problem, but we note that seen as a function in T, the values of the process v are infinite-dimensional curves.

Variance Curve Functionals

Such an infinite-dimensional object is both hard to handle in applications and also unrealistic: In fact, only a finitely many variance swap maturities are traded. Hence, at time $t = 0$, we actually want to *interpolate* the market prices of variance swaps by a finite-dimensionally parameterized non-negative functional G:

$$V_0(T) = \int_0^T G(Z_0; x)\, dx$$

where $Z_0 \in \mathbb{R}^m$ is some parameter vector.[2] Now assume that we can find a finite-dimensional diffusion process Z such that

$$v_t(T) = G(Z_t; T - t) \tag{1}$$

interpolates the observed market prices well. We then call G a *variance curve functional* and Z its *parameter process.*

Note that G represents the curve in terms of fixed time-to-maturity $x := T - t$ rather than in maturity T. Such a Musiela-representation [21] is advantageous for two reasons: First, the characteristics of a fixed time-to-maturity swap are more stable than those of fixed maturity swaps and therefore easier to assess statistically. Secondly, and more importantly, the above representation properly describes $\varsigma_t := v_t(t)$ as a predicable and integrable process. We can hence use it to define the *associated stock price process*

$$S_t := \mathcal{E}_t \left(\int_0^{\cdot} \sqrt{\varsigma_s}\, dB_s \right) \tag{2}$$

if (1) is used to *define* a variance curve (the symbol $\mathcal{E}$ denotes the Dolean-Dade-exponential and B some standard Brownian motion).

[2] In practise, one such functional G is given implicitly if Neuberger's [22] method to compute variance swap prices is applied: The volatilities of the underlying vanilla prices are usually given by an interpolation of the implied volatilities.

Writing the diffusion Z as $dZ_t = \mu(t;Z_t)\,dt + \sigma(t;Z_t)\,dW_t$, the task at hand is to assess whether G and Z are *consistent* in the sense that v_t given by (1) is a non-negative martingale. Some basic calculus shows (theorem 2.1) that (G,Z) are consistent if and essentially only if

$$\partial_x G(Z_t;x) = \mu(t;Z_t)\,\partial_z G(Z_t;x) + \frac{1}{2}\sigma^2(t;Z_t)\,\partial_{zz} G(Z_t;x)\ . \tag{3}$$

We also show that if the forward variance process is then defined by (1), the associated index price process with instantaneous variance $\varsigma_t = v_t(t)$ is indeed a local martingale and the market with the index price process (2) and the price processes of all variance swaps is free of arbitrage.

PROBLEM *Find realistic functionals G and, for a given G, a consistent diffusion process Z such that (3) holds.*

We will present a few such curve functionals in section 3. One example is the functional

$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x} \qquad (z_1, z_2, z_3) \in {\mathbb{R}^+}^3\ .$$

This functional is a first non-trivial representant from the class of exponential-polynomial models which have been discussed by Filipovic [13] for interest rates. Similar to him, we find that in general the coefficients in the exponentials (ie, z_3 above) must be constant if the curve is to produce arbitrage-free variance swap prices.

We also find that none of three investigated standard implied volatility term-structure functionals such as "square root in time" can be used as a variance curve functional.

We then combine those results and apply them to the standard market practise of recalibration of various models. Taking Heston's popular model [19] as an example, we show that mean-reversion must be kept constant during the life of an exotic to ensure that its price process is a local martingale *in the real world of the institution* (where it is the result of frequent recalibration).

We also show that the abovementioned implied volatility term-structure functionals we cannot consistently recalibrated to the market.

Structure of the article

We will start in the next section by introducing the main assumptions and then continue with the build-up of the variance swap term-structure, the definition of the forward variance curve and the derivation of the corresponding HJM-type conditions. An application of Hilbert space calculus shows how we can simplify matters when we switch from a parameterization in fixed maturity T to fixed time-to-maturity x.

In the third section, we will then present some examples. The fourth section will review the impact of our findings on the practise of daily re-calibration.

We conclude in section five.

2 Variance Curve Models

We want to discuss variance curve models, where we model the stock price process and the full term structure of variance swaps, or rather the forward variance curve.

2.1 The Model Framework

We will model a continuous positive index price process S which pays no dividends in a market where the prevailing interest rates are zero. We want to model it as a local martingale on a stochastic base $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{P}, \mathbb{F})$ with an n-dimensional Brownian motion $W = (W^i)_{i=1,\dots,n}$ and which creates the filtration $\mathbb{F}$. Following Revuz/Yor [25] chapter IV.2 we denote by H^2 the space of all square-integrable martingales, L^2 the space of all predicable processes φ such that $\mathbb{E}[\int_0^T \varphi_s^2\, ds] < \infty$ and H^{loc} and L^{loc} the respective local spaces. The symbol $\mathcal{P}$ denotes the previsible σ-algebra on $\Omega \times [0, \infty)$ and $\mathcal{B}(V)$ is the Borel-σ-algebra of a topological space V. We will use the symbols $x \vee y := \max(x, y)$ and $x \wedge y := \min(x, y)$.

The financial contract which pay out $\langle \log S\rangle_T$ at T is called a (zero mean) *variance swap*. We denote the price of such a contract at time t by $V_t(T)$ with $V_t(T) = V_T(T)$ for $t > T$.

Definition 2.1 (Variance Swap model) *We call the pair (S, V) with $V = (V(T))_{T\in[0,\infty)}$ a* Variance Swap Model *(on $(\mathbb{P}, \mathbb{F})$) if*

1. *The price process S is a positive continuous local martingale,*
2. *for each $T < \infty$, the process $V(T)$ is a non-negative martingale with $V_0(T) < \infty$ and $V_T(T) = \langle \log S\rangle_T$.*

The above conditions that all tradables are local martingales on $(\mathbb{P}, \mathbb{G})$ are generally equivalent to "No Free Lunch with Vanishing Risk" as shown by Delbaen / Schachermayer [8].

Let (S, V) a variance swap model. Since S is a positive local martingale, we can write it as

$$S_t = \mathcal{E}_t(X) \quad \text{with} \quad X \in H^{\text{loc}}$$

(without loss of generalization we set $S_0 = 1$). Because the filtration $\mathbb{F}$ is generated by W we find some non-negative *short variance* process $\varsigma \in H^{\text{loc}}$ and a "driving" Brownian motion B adapted to $\mathbb{F}$ such that

$$X_t = \int_0^t \sqrt{\varsigma_s}\, dB_s \ .$$

Then we have by construction

$$\langle \log S\rangle_t = \langle X\rangle_t = \int_0^t \varsigma_s\, ds \ .$$

Since the price processes $V(T)$ are given as

$$V_t(T) = \mathbb{E}\left[\langle \log S\rangle_T \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^T \varsigma_s\, ds \,\middle|\, \mathcal{F}_t\right] \tag{4}$$

we obtain $\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\langle X\rangle_T\right] = V_0(T) < \infty$ and therefore

PROPOSITION 2.1 *X is a square-integrable martingale up to each T.*

The martingales $V_t(T) := \mathbb{E}[\int_0^T \varsigma_s\, ds | \mathcal{F}_t]$ admit a representation

$$V_t(T) = V_0(T) + \int_0^t b_s(T)\, dW_s \qquad b(T) \in L^{\text{loc}}\ .$$

and are absolutely continuous in T (since Fubini applies) with derivative

$$v_t(T) := \partial_T V_t(T) = \mathbb{E}\left[\varsigma_T \,\middle|\, \mathcal{F}_t\right]\ . \tag{5}$$

We call $v(T)$ the (fixed maturity) *forward variance* (compare the similarity with the forward rate in interest rate modelling). With (5), we have also shown:

PROPOSITION 2.2 (HJM condition for Forward Variance Curves I) *If (S, V) is a variance swap model, then the process $v_t(T) := \partial_T V_t(T)$ is a martingale and can be written as*

$$v_t(T) = v_0(T) + \int_0^t \beta_s(T)\, dW_s \tag{6}$$

with $\beta(T) := \partial_T b(T) \in L^{\text{loc}}$.

The fact that $\beta(T) = \partial_T b(T)$ follows from the uniqueness of the representation of a local martingale on $\mathbb{F}$ with respect to W.

Now consider that v is given and that we want to use these processes to define a variance swap model:

DEFINITION 2.2 (Variance Curve Model) *We call $v = (v(T))_{T\in[0,\infty)}$ a* Variance Curve Model *(on $\mathbb{W}$) if for all $T < \infty$:*

1. *$v(T)$ is a non-negative continuous martingale with representation*

$$dv_t(T) = \beta_t(T)\, dW_t$$

   for some $\beta(T) \in L^{\text{loc}}$ and

2. *$V_0(T) := \int_0^T v_0(x)\, dx$ exists and is finite.*

For any such model and a real-valued Brownian motion B, the associated stock price process *is then defined as the local martingale*

$$S_t := \mathcal{E}_t\left(\int_0^\cdot \sqrt{\varsigma_t}\, dB_t\right)$$

and (S, V) is a variance swap model.

Proof – First of all note that $\varsigma_t := v_t(t)$ is well-defined since it is continuous in both t and T. Moreover, $\mathbb{E}[\int_0^T \varsigma_s \, ds] = \int_0^T \mathbb{E}[\varsigma_t(t)]dt = \int_0^T \mathbb{E}[\varsigma_0(t)]dt = V_0(T) < \infty$, i.e. $\sqrt{\varsigma} \in L^2$. Hence, S is a local martingale.

It remains to show that $V_t(T) := \int_0^T v_t(s) \, ds$ is a martingale. The process $V(T)$ is clearly adapted. Moreover, $\mathbb{E}[V_T(T)] = \mathbb{E}[\int_0^T v_T(x) \, dx] = \int_0^T \mathbb{E}[v_T(x)] \, dx = V_0(T) < \infty$, so $V(T)$ is indeed a martingale. $\square$

We have seen that a variance curve model like an HJM interest-rate model is full determined by specifying the "initial curve" v_0, the "volatility structure" β and the "correlation structure" B.

REMARK 2.1 *Dupire [12] has used a similar result as proposition 2.2 and applied it to a representation of the forward variance as an exponential, i.e.*

$$v_t(T) = v_0(T)\mathcal{E}\left(\int_0^t \beta_s^*(T) \, dW_s\right) \tag{7}$$

See also corollary 2.1 in the case of finite-dimensional representations.

2.1.1 Fixed time-to-maturity

In the above definition 2.2, we need $v_t(t)$ to define the associated stock price process. In the current setting, this object is defined across the maturities T. Let us consider therefore:

DEFINITION 2.3 *We call*

$$\hat{v}_t(x) := v_t(t + x)$$

the fixed time-to-maturity forward variance*, and* $\hat{V}_t(x) := \int_0^x \hat{v}_t(s) \, ds$ *the* fixed time-to-maturity variance swap.

REMARK 2.2 *Even though such a fixed time-to-maturity parameterization for HJM-models has been introduced by Musiela [21], it is more common in interest-rate theory to deal with fixed maturity objects because the maturities of underlying market instruments are typically fixed points in time (such as LIBOR rates and Swaps).*[3]

A variance curve, in contrast, is more naturally seen as a fixed time-to-maturity object, in particular given that the short end of the curve is the instantaneous variance of the log-price of the stock.

Note that above definition is valid for each fixed t and almost all ω. To define a proper process $\hat{v}$, we have to impose some additional regularity on v:

[3] In a typical LIBOR rate model, the short rate is not modelled.

Proposition 2.3 (HJM condition for Forward Variance Curves II) *Let (S, v) be a variance curve model. Assume that v_0 is differentiable in T, that β in (6) is $\mathcal{B}[\mathbb{R}] \times \mathcal{P}$-measurable and almost surely differentiable in T with*

$$\sqrt{\int_0^\infty \partial_T \beta_t(x)^2 \, dx} \in L^{\mathrm{loc}} \ . \tag{8}$$

Then, $\partial_T v_t(T)$ is given as

$$\partial_T v_t(T) = \partial_T v_0(T) + \int_0^t \partial_T \beta_s(T) \, dW_s \ , \tag{9}$$

(see Protter [24] p. 208) and the fixed time-to-maturity forward variance $\hat{v}(x)$ has the form

$$\hat{v}_t(x) = \hat{v}_0(x) + \int_0^t \partial_x \hat{v}_s(x) \, ds + \int_0^t \hat{\beta}_s(x) \, dW_s \tag{10}$$

where $\hat{\beta}_t(x) := \beta_t(t + x)$.

Proof – With the assumptions above, we have

$$\begin{aligned}
\hat{v}_t(x) &= v_t(t+x) \\
&\overset{(6)}{=} v_0(t+x) + \int_0^t \partial_T \beta_s(u+x) \, dW_u \\
&\overset{v_0, \beta_u \in C^1}{=} v_0(x) + \int_0^t \partial_T v_0(s+x) \, ds \\
&\qquad + \int_0^T \left\{ \beta_u(u+x) + \int_u^t \partial_T \beta_u(s+x) \, ds \right\} dW_u \\
&\overset{(8)}{=} v_0(x) + \int_0^t \left\{ \partial_T v_0(s+x) + \int_0^s \partial_T \beta_u(s+x) \, dW_u \right\} ds \\
&\qquad + \int_0^T \beta_u(u+x) \, dW_u \\
&\overset{(9)}{=} v_0(x) + \int_0^t \partial_T v_s(s+x) \, ds + \int_0^T \beta_u(u+x) \, dW_u \\
&= \hat{v}_0(x) + \int_0^t \partial_T \hat{v}_s(x) \, ds + \int_0^T \hat{\beta}_u(x) \, dW_u \ ,
\end{aligned}$$

as claimed. Note that (8) basically ensures that $\int_0^s \partial_T \beta_u(T) \, dW_u$ is a local martingale. □

From (10) it is natural to extend the framework to model $\hat{v}$ as a martingale with values in an Hilbert space. Details will be carried out in section 2.3.

For practical implementations, however, it is more suitable to represent v and $\hat{v}$ via finite-dimensional diffusion processes.

2.2 Consistent Variance Curve Functionals

Consider the situation in the reality of a trading floor: We do not actually see an infinite number of variance swap prices $(V_0(T))_{T\in[0,\infty)}$ in the market. Rather a discrete set of swap prices will be *interpolated* by some functional, which is parameterized by a finite-dimensional parameter vector.

DEFINITION 2.4 (Variance Curve Functional) *A* Variance Curve Functional *with* parameter space $\mathcal{Z} \in \mathbb{R}^m$ *is a non-negative* $C^{2,1}$*-function*[4] $G : (z; x) \in \mathcal{Z}\times\mathbb{R}_0^+ \to \mathbb{R}_0^+$ *such that* $\int_0^T G(z; x)\, dx < \infty$ *for all* (z, T).

Given a functional G, we now have to find a process $Z = (Z_t)_{t\in[0,\infty)}$ of parameters such that

$$\hat{v}_t(x) := G(Z_t; x)\ , \quad x \geq 0\ ,$$

forms a variance curve model. To avoid arbitrage, we face a situation similar to the "consistency problem" described by Björk and Christensen [3]: Given G, can we find a process Z such that v is a martingale?

DEFINITION 2.5 (Consistent Parameter Process) *A* Consistent Parameter Process *for* $(G, \mathcal{Z})$ *is a continuous diffusion process* Z*, such that for all* $Z_0 \in \mathcal{Z}$ *we have* $Z_t \in \mathcal{Z}$ *and the process*

$$v_t(T) := G(Z_t; T - t)\ , \quad t \leq T\ ,$$

is a martingale. In this case, v *is a variance curve model and we call* Z *and* G consistent.

REMARK 2.3 *Note that*

$$\varsigma_t = G(Z_t; 0)$$

is the instantaneous variance of an associated stock price process of the variance curve model given by (G, Z)*. Here,* $\sqrt{\varsigma} \in L^2$ *by construction. In this sense,* (G, Z) *defines a variance swap model.*

Naturally, we now have to ask whether a given curve functional G is consistent at all and if so, whether we can specify a particular consistent process. Let

$$dZ_t = \mu(t; Z_t)\, dt + \sigma(t; Z_t)\, dW_t\ . \tag{11}$$

We will want to chose (μ, σ) such that a unique solution exists. For example, local Lipschitz-continuity is sufficient.

NOTATION *Let* $\Xi = \Xi(\mathcal{Z})$ *be the set of processes* Z *with coefficients* (μ, σ) *such that* Z *is a unique strong solution of (11) for all* $Z_0 \in \mathcal{Z}$ *and does not explode in finite time. We also write* $(\mu, \sigma) \in \Xi$ *and denote by* Ξ^h *the subset of time-homogeneous* (μ, σ).

[4] Differentiability relates to the relative interior of $\mathcal{Z}$.

The following theorem is just a simple application of Ito's formula (compare also proposition 3.1.1 in [13]).

THEOREM 2.1 (HJM condition for Variance Curve Functionals) *A process $Z \in \Xi$ is consistent with $(G, \mathcal{Z})$ if and only if*

$$\left.\begin{aligned} \partial_x G(Z_t; x) = \mu(t; Z_t)\, \partial_z G(Z_t; x) + \tfrac{1}{2}\sigma^2(t; Z_t)\, \partial_{zz} G(Z_t; x) \\ Z_t \in \mathcal{Z} \end{aligned}\right\} \tag{12}$$

holds $\mathbb{P} \otimes \lambda|_{[0,\infty)}$-as for all $Z_0 \in \mathcal{Z}$. Here, we understand as usual $\mu\, \partial_z G_t = \sum_j \mu_j\, \partial_{z^j} G_t$ and $\sigma^2\, \partial_{zz} G = \sum_{i,j} \sigma_i \sigma_j\, \partial_{z^i z^j} G$.

Proof – First assume Z is consistent with G. Then, (12) holds $\mathbb{P} \otimes \lambda|_{[0,\infty)}$-as by application of Ito's formula to since $v_t(T) := G(Z_t; T-t)$ is a martingale.

Now assume Z is such that (12) holds. Then $v_t(T) := G(Z_t; T-t)$ for $t \le T$ is a local martingale by Ito's formula. Moreover, we have $v_T(T) = G(Z_T; 0)$, hence $v_t(T) = \mathbb{E}\left[\, G(Z_T; 0) \,\middle|\, \mathcal{F}_t \,\right]$ and v must be a martingale. □

Now note that (12) is an equation in (Z_t, x) which holds $\mathbb{P} \otimes \lambda|_{[0,\infty)}$ almost surely. However, Z_t might have a much smaller support than $\mathcal{Z}$.

DEFINITION 2.6 *We will call a process $Z \in \Xi$* essentially diffuse *if there exists a* state space *$\mathcal{D} \subset \mathbb{R}^m$ such that for almost all $s \ge 0$, $z_s \in \mathcal{D}$ and all $t > s$:*

1. *The support of Z_t started in $Z_s = z_s$ is $\mathcal{D}$.*
2. *We have $\lambda^m \ll \mathbb{P}[Z_t \in \cdot | Z_s = z_s]$ on $\mathcal{D}$.*

We denote by Ξ^* the subset of such processes in Ξ.

This definition excludes singular unconnected points to which the process would have to jump to (such as $[0,1] \cup \{2\} \subset \mathbb{R}^+$) and state spaces where one component is constant or time-dependent, for example $\operatorname{supp}(Z_t) = \mathbb{R}^+ \times \{\alpha t\} \subset \mathbb{R}^2$. It also excludes "branching processes" which change their direction in certain reflection points or absorbing states.

COROLLARY 2.1 (HJM condition as PDE) *An essentially diffuse process $Z \in \Xi^*$ is consistent with G if and only if $\mathcal{D} = \mathcal{Z}$ and*

$$\partial_x G(z; x) = \mu(t; z)\, \partial_z G(z; x) + \frac{1}{2}\sigma^2(t; z)\, \partial_{zz} G(z; x) \tag{13}$$

holds λ^{m+2}-as on $\mathcal{D} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$.

Proof – Fix $s \ge 0$ and $z_s \in \mathcal{Z}$. Since Z is essentially diffuse, let $\mathbb{P}[Z_t \in \cdot | Z_s = z_s] = p_t + o_t$ where $p_t \approx \lambda^m|_{\mathcal{D}}$ and $o_t \perp \lambda^m$. Let ψ_t be the strictly positive density of p_t wrt to $\lambda^m|_{\mathcal{D}}$. Then, for any appropriate F,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}\otimes\lambda}\left[\, F(t; Z_t) \,\middle|\, Z_s = z_s \,\right] &= \int_{\mathbb{R}^+} \int_{\mathcal{D}} F(t, z) \mathbb{P}[Z_t \in dz | Z_s = z_s]\, dt \\ &\ge \int_{\mathbb{R}^+} \int_{\mathbb{R}^m} F(t, z) 1_{z \in \mathcal{D}}\, \psi_t(z)\, dz\, dt \end{aligned}$$

Hence, if (12) holds $\mathbb{P} \otimes \lambda$-as then (13) holds λ^{m+1}-as on $\mathcal{D} \times \mathbb{R}^+$. The reverse assertion follows since Z_t takes only values in $\mathcal{D}$. □

A curve functional is of the form $G(z;x) = \mathbb{E}\left[\, G(Z_{t+x};0) \,\middle|\, Z_t = z \,\right]$. This representation is independent of t, and the process Z can indeed often assumed to be homogeneous: For example, if (μ, σ) are locally Lipschitz, then let $\mathcal{U}$ be the set of $t \geq 0$ such that equation (13) is satisfied in (z,t,x) for almost all $(z,x) \in \mathcal{Z} \times \mathbb{R}_0^+$. Fix some $u \in \mathcal{U}$. Then, the homogeneous coefficients $(\mu(u,\cdot), \sigma(u,\cdot))$ are also Lipschitz and the SDE in t,

$$dZ_t^u = \mu(u; Z_t^u)\, dt + \sigma(u; Z_t^u)\, dW_t \quad Z_0^u = Z_0 \tag{14}$$

admits a strong solution for all $Z_0 \in \mathcal{Z}$ and is time-homogeneous. By definition, Z^u is also consistent with G.

If G itself is a positive transformation of some other function g, we can translate corollary 2.1 accordingly:

Corollary 2.2 *Let H be a positive smooth function and $G(z;x) := H(g(z;x))$ such that G is a variance curve functional. With the conditions of corollary 2.1, Z is consistent with G if and only if*

$$H'(g)\partial_x g = \mu\, H'(g)\partial_z g + \frac{1}{2}\sigma^2 \left\{ H''(g)(\partial_z g)^2 + H'(g)\partial_{zz} g \right\} \tag{15}$$

on $\mathcal{D} \times \mathbb{R}^+$ and $\int_0^T G(z;x)\, dx < \infty$ for all $T < \infty$. In particular,

- *In case $H(g) := \exp(g)$, then (15) reduces to*

$$\partial_x g = \mu\, \partial_z g + \frac{1}{2}\sigma^2 \left\{ (\partial_z g)^2 + \partial_{zz} g \right\} \ . \tag{16}$$

- *For $H(g) := \frac{1}{2}g^2$, we get*

$$g\partial_x g = \mu\, g\partial_z g + \frac{1}{2}\sigma^2 \left\{ (\partial_z g)^2 + g\partial_{zz} g \right\} \ .$$

Equation (16) is the relevant condition if we want to deal with exponential models such as (7).

Remark 2.4 (Hedging with Variance Swaps) *We have built our variance curve model on a stochastic space where the measure $\mathbb{P}$ was fixed and a martingale measure. In general, of course, this measure does not need to be unique. However, if the process $\mathcal{S} = (X, v(T)_{T\in[0,\infty)})$ is extremal on its own filtration, the model becomes complete and we can hedge arbitrary exotic payoffs with finitely many variance swaps and the stock. Whether $\mathcal{S}$ is extremal on its filtration essentially depends on invertibility of its volatility matrix in an appropriate sense (see Karatzas/Shreve [20] chapter 6.7).*

Remark 2.5 *A consistent variance curve model will typically have remaining free parameters. In practical applications, the prices of European options on the underlying stock can be used to calibrate these parameters, such that the resulting model fits both the variance swap and the vanilla option market.*

2.3 Infinite-Dimensional Models

As promised in the end of section 2.1.1, we now show how to apply the framework of Hilbert space stochastic calculus to variance curves in infinite dimensions. In particular, we will construct a Hilbert space H^+ such that a result similar to proposition 2.3 can be derived under the alternative assumption that the variance curves are martingales with elements in H^+ (proposition 2.5). Moreover, we will show how the results of the previous sections translate into this setting.

We first recall the main facts on stochastic calculus in Hilbert spaces. The reader is referred to Filipovic [13] or DaPrato/Zabzyk [23] for more details: To allow application of the results in [13], we will need a convenient Hilbert space in which our variance curve processes will take their values.

Define as in [13] p. 75ff a norm for absolutely continuous functions h via

$$||h||_H^2 := |h(0)|^2 + \int_0^T |h'(s)|^2 \, w(s) \, ds$$

for some non-decreasing weighting function w. We will use $w(s) := e^{\alpha s}$ for $\alpha > 0$. Then,

$$H := \left\{ \; h(x) = \int_0^x h'(s) \, ds \;\middle|\; ||h||_H < \infty \; \right\} \; .$$

is a separable Hilbert space (theorem 5.1.1 in [13]) with scalar product

$$< h, g >_T = h(0)g(0) + \int_0^T h'(s)g'(s)w(s) \, ds$$

Note that by definition $h \in H$ has a continuous version with finite limit $\lim_{x\to\infty} h(x)$ (see [13] p. 77). Accordingly, h is bounded. The space H satisfies

- (H1) The functions $h \in H$ are continuous and $h(x)$ is a continuous linear functional.
- (H2) The semigroup $S(t)$ of right-shifts given by $(S(t)h)(x) := h(t + x)$ is strongly continuous in H with infinitesimal generator A such that $Ah = h'$.

Also see theorem 5.1.1 in [13] and remark 5.1.1 (with regards to the fact that we do not require that $w^{-1/3} \in L^1(\mathbb{R}^+)$). As a consequence, $\int_0^{\cdot} h(y) \, dy$ is continuous linear functional on H, see lemma 4.3.1 in [13]. Hence,

Proposition 2.4 *We have $|| \int_0^x h(y) \, dy||_H \leq xu(x)$ for some constant u which depends only on x.*

We will consider positive functions in H as models for the (forward) variance curve. Hence, let

$$H^+ := \{ \ h \in H \ : \ h \geq 0 \ \lambda\text{-almost surely} \ \} \ .$$

Armed with this Hilbert space, we will now briefly recall how we can define a Brownian motion integrands such that the resulting martingales take values in our Hilbert space H.

Let U be a second Hilbert space, for example the canonical Hilbert space of sequences ℓ^2 as in Filipovic [13] p. 14). Let $(W^i)_{i=1,\dots}$ be the countable sequence of independent standard Brownian motions defined on $\mathbb{W}$. We fix an orthonormal basis $(g_i)_{i=1,\dots}$ of an extended space (see Da Prato/Zabzyk [23] p. 96) such that

$$\sum_{i=1}^{\infty} g_i W_t^i$$

defines what we call U-valued cylindrical Brownian motion (with respect to an operator Q which guarantees that W exists; see [13] p.14). We will abuse notation by also writing W for this process.

Note since we want to integrate over W such that the result are elements of the Hilbert space H, the integrands must be linear operators $\varphi : U \to H$. We introduce the operator norm

$$||\varphi||^2_{L_2^0(H)} := \sum_{i=1}^{\infty} ||\varphi(g_i)||_H^2$$

and define $L_2^0(H)$ as the space of those operators for which this norm is finite. Now denote by $\mathcal{L}_T^2(H)$ the space of (equivalence classes of) predictable processes ϕ with values $\phi_t \in L_2^0(H)$ such that

$$||\phi||^2_{\mathcal{L}_T^2(H)} := \mathbb{E}\left[\int_0^T ||\phi_t||^2_{L_2^0(J)} \right]$$

is finite. Using the usual approximation via simple integrands we can then define the integral

$$\int_0^t \phi_s \, dW_s$$

such that the intuitive relation

$$\int_0^t \phi_s \, dW_s \xleftarrow{\mathbb{P}} \sum_{i=1}^{\infty} \phi_s(g_i) \, W_t^i$$

holds (see [13] proposition 2.2.1 p. 17). Such an integral defines a square-integrable martingale with respect to $||M||^2_{\mathcal{H}_T^2(H)} := \mathbb{E}[||M_T||_H^2]$, if $\phi \in \mathcal{L}_T^2(H)$.

As usual, we can then extend the space via localization: If we consider the space $\mathcal{L}_T^{\text{loc}}(H)$ of predictable processes ϕ with values in $L_2^0(H)$ such that only $\mathbb{P}[\int_0^T ||\phi_t||^2_{L_2^0(H)}] < 1$ holds, then we obtain the space $\mathcal{H}_T^{\text{loc}}(H)$ of local martingales. We will omit the notion of T if a process is in $\mathcal{H}_T^{\text{loc}}$ for all T and define $\mathcal{H}^1$ as the space of all martingales. A subscript"+" on $\mathcal{H}$ will denote non-negative processes.

More generally, we call a pair (h, g) of predictable processes *integrable* if

$$\mathbb{P}\left[\int_0^T \left(||h_s||_H + ||g_s||^2_{L_2^0(H)}\right) ds < \infty\right] \tag{17}$$

In that case, $M_t := \int_0^t h_s\, ds + \int_0^t g_s\, dW_s$ is a semimartingale. This condition resembles (C2) of Filipovic [13] p. 59. We will also assume (C1), that is the initial curves are always elements of H^+.

REMARK 2.6 *In this setting it is also true that if the tradables are local martingales then the market is free of arbitrage in the strong sense that it does not admit a "free lunch with vanishing risk".*

2.3.1 Variance Curve Processes in Infinite Dimensions

After we have now clarified the setting, we will limit our attention to forward variance curve processes v which take values in H^+. To this end, assume that $\mathbb{W}$ supports countably many independent Brownian motions $W = (W^i)_{i=1,\dots}$. The relatively unhandy assumptions in proposition 2.3 can then be replaced by:

ASSUMPTION 1 *We assume that $v \in \mathcal{H}_+^{\text{loc}}$.*

Since v is a local martingale, we have in particular

$$dv_t(T) = b_t(T)\, dW_t$$

where W is the U-valued Brownian motion introduced above. Let as before $\hat{v}_t(x) := v_t(t + x)$. Then, we obtain:

PROPOSITION 2.5 *For all $v_t \in \mathcal{H}_+^{\text{loc}}$, we also have $\hat{v}_t \in \mathcal{H}_+^{\text{loc}}$ with*

$$d\hat{v}_t(x) = \partial_x \hat{v}_t(x)\, dt + \hat{b}_t(x)\, dW_t \tag{18}$$

where $\hat{b}_t(x) := b_t(t + x)$.

Proof – Recall that $S(u)v_t(\cdot) = v_t(u + \cdot)$. Then,

$$\begin{aligned}\hat{v}_t(x) &= v_t(t + x) \\ &= v_0(t + x) + \int_0^t b_s(t + x)\, dW_s\end{aligned}$$

$$
\begin{aligned}
&= S(t)\hat{v}_0(x) + \int_0^t S(t-s) b_s(s+x)\, dW_s \\
&= S(t)\hat{v}_0(x) + \int_0^t S(t-s)\hat{b}_t(s)\, dW_s \ .
\end{aligned}
$$

Using Filipovic [13] p. 24 we find that $\hat{v}_t(x)$ is then a mild solution to the linear equation

$$
d\hat{v}_t = A\hat{v}_t\, dt + \hat{b}_t\, dW_t \tag{19}
$$

where A is the infinitesimal operator of S. Since $A\hat{v}_t = \partial_x \hat{v}_t$ (see section 2.3) this shows in particular that $(\partial_x \hat{v}, \hat{b})$ is integrable. □

The additional assumption 1 above turns out to be relatively strong:

LEMMA 2.1 *Assumption 1 implies that proposition 2.3 holds.*

Proof – We first have to prove that V is a martingale for any $v \in \mathcal{H}_+^{\text{loc}}$.

To this end, note that the process $\bar{V}(T)$ is adapted by construction. Due to proposition 2.4, we have that $||V_t(T)||_H \leq Tu(T) < \infty$, hence $||V(T)||^2_{\mathcal{H}^2_T} = \mathbb{E}[||V_T(T)||^2_H] < \infty$, even if v is only a local martingale. The martingale property then follows because $v(T)$ is a martingale for all T.

As for proposition 2.1, it follows then that X is a square-integrable martingale and S is therefore a local martingale. □

A simple application of Ito's lemma also yields a similar statement as theorem 2.1 for the case of functionals which take values in H^+.

PROPOSITION 2.6 *Any function G with values in H^+ which admits a parameter process such that (12) holds is a consistent variance curve Functional, i.e.*

$$
v_t(T) = G(Z_t; T-t) \ .
$$

The Hilbert space approach is more suitable if we are to model the variance curve as an infinite-dimensional object, since it is not anymore necessary to impose additional constraints on β. These are well embedded in condition (H2) on H (compare p. 59 in [13]), which also ensures that $\hat{v}_t(0)$ is properly defined.

3 Examples of Variance Curve Models

In this section we present some examples. We will discuss variance curve models which can be reduced to standard stochastic volatility models and will show that various standard implied volatility term structure schemes are not consistent in the sense developed above.

3.1 Exponential-Polynomial Variance Curve Models

The family of *Exponential-Polynomial Curve Funtional* is parameterized by $z = (z_1, \ldots, z_m; z_{m+1}, \ldots, z_d) \in \mathbb{R}^{+m} \times \mathbb{R}^{d-m}$ and given as

$$G(z;x) = \sum_{i=1}^{m} p_i(z;x) e^{-z_i x} \tag{20}$$

where p_i are polynomials of the form $p_i(x) = \sum_{j=0}^{N} a_{ij}(z) x^j$ with coefficents a such that $p_i > 0$ λ^d-as, also compare Filipovic [13]. We assume that any parameter process has $Z_t^i \neq Z_t^j$ for $i \neq j$, since otherwise we can just rewrite (20) accordingly. Also note that $\int_0^T G(z;x)\, dx < \infty$ for all $T < \infty$.

Lemma 3.1 *The coordinates $Z^1, \ldots, Z^m$ of any consistent parameter process Z are constant.*

Proof – We have

$$\begin{aligned}
\partial_x G &= -z_i \sum_{i=1}^{m} p_i(z;x) e^{-z_i x} + \sum_{i=1}^{m} \partial_x p_i(z;x) e^{-z_i x} \\
\partial_{z_j} G &= -p_i(z;x) x e^{-z_i x} 1_{j \le m} + \sum_{i=1}^{m} \partial_{z_j} p_i(z;x) e^{-z_i x} \qquad (21) \\
\partial^2_{z_j z_j} G &= \left(p_i(z;x) x^2 e^{-z_i x} - 2 \partial_{z_j} p_i(z;x) x e^{-z_i x} \right) 1_{j \le m} + \sum_{i=1}^{m} \partial^2_{z_j z_j} p_i(z;x) e^{-z_i x}
\end{aligned}$$

We ignore the mixed terms $\partial^2_{z_j z_k} G$, since we can already see that $\partial^2_{z_j z_j} G$ with $j \le m$ are the only terms in (13) which involve polynomials of degree grad$p_i + 2$ as factors in front of the exponentials $e^{-z_i x}$. Since we choose the z_i distinct, and because neither μ nor σ depends on x, this implies that $\sigma_i^2 = 0$ for $i \le m$. In other words, the states z_i for $i \le m$ cannot be random.

Next, we use (21) and find with the same reasoning (now applied to the polynomials of degree gradp_i+1) that $\mu_i = 0$ for $i \le m$, so Z^i must be a constant. □

We will now present two particular exponential-polynomial curve functionals. In the light of lemma 3.1, we will keep the exponentials constant but investigate the possible dynamics of the remaining parameters.

Example 1 (Linearly Mean-Reverting Variance Curve Models) *The Functional*

$$G(z;x) := z_2 + (z_1 - z_2) e^{-\kappa x} \ .$$

with $z \in \mathbb{R}^+ \times \mathbb{R}+$ is consistent with Z if $\mu_1(t;z) = \kappa(z_2 - z_1)$ and $\mu_2(t;z) = 0$ (that is, Z^2 must be a martingale). The volatility parameters can be freely specified, as long as Z^1 and Z^2 remain non-negative.

We call such a model a linearly mean-reverting variance curve model.

A popular parametrization is $\sigma_2 = 0$ and $\sigma_1(t; z_1) = \nu\sqrt{z_1}$ for some $\nu > 0$, which has been introduced by Heston [19]. (Z^1 is then the square of the short-volatility of the associated stock price process). More generally, we can set $\sigma_1(t; z_1) = \nu z_1^\alpha$ for some constant $\alpha \geq \frac{1}{2}$. See Andersen/Piterbarg [1] for some more details on such models.

This example also shows the reason why we allow for inhomogeneous processes: The volatility process of Z can well be specified with some term-structure, e.g. to take into account seasonal effects like lower volatility over different trading periods.

Proof – Corollary 2.1 implies that we have to match

$$-\kappa(z_1 - z_2)e^{-\kappa x} \stackrel{!}{=} \mu_1(t; z)e^{-\kappa x} + \mu_2(t; z)(1 - e^{-\kappa x})$$

Since the left hand side has no term constant in x, we must have $\mu_2(t; z) = 0$ (i.e. Z^2 is martingale), and then $\mu_1(t; z) = \kappa(z_2 - z_1)$. □

The next model is a generalization of the above. We will omit the proof which works similar as above.

Example 2 (Double Mean-Reverting Variance Curve Models) *Let $c, \kappa > 0$ constant and let $z = (z_1, z_2, z_3) \in \mathbb{R}^{+3}$. The Curve Functional*

$$G(z; x) := \begin{cases} z_3 + (z_1 - z_3)e^{-\kappa x} + (z_2 - z_3)\kappa \frac{e^{-cx} - e^{-\kappa x}}{\kappa - c} & (\kappa \neq c) \\ z_3 + (z_1 - z_3 - \kappa x(z_2 - z_3))\, e^{-\kappa x} & (\kappa = c) \end{cases} \tag{22}$$

is consistent with any parameter process Z such that

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)\, dt + \sigma_1(Z_t)\, dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)\, dt + \sigma_2(Z_t)\, dW_t \\ dZ_t^3 &= \sigma_3(Z_t)\, dW_t \end{aligned}$$

and is called a double mean-reverting variance curve model.

This turns out to be a flexible and applicable model: At the time of writing, the variance functional (22) fits the variance swap market of major indices well, so this kind of double mean-reverting models is a good candidate for a variance curve model. Such a model has recently been successfully implemented to price and hedge options on variance and related products. Higher order models with similar structure can also be considered.

3.2 Exponential Curve Models

As in (20), let $(p_i)_{i=1,\dots,m}$ be polynomials and let

$$g(z; x) = \sum_{i=1}^{m} p_1(z; x)e^{-z_i x}$$

with $(z_1, \ldots, z_m) \in \mathbb{R}^{+m}$. Set

$$G(z; x) := \exp(g(z; x)) \ . \tag{23}$$

Using corollary 2.2, we find similar to the above results (also compare theorems 3.6.1 and 3.6.2 from Filipovic [13] p.52ff):

LEMMA 3.2 *For each consistent parameter process Z for G, the coordinates $Z^1, \ldots, Z^m$ are constant. Moreover, there must be at least one pair $i \neq j$ such that $Z^i = 2Z^j$, otherwise Z is entirely constant.*

EXAMPLE 3 (Exponential Mean-Reverting Models) *Let*

$$g(z; x) = z_2 + (z_1 - z_2)e^{-\kappa x} + \frac{z_3}{4\kappa}(1 - e^{-2\kappa x})$$

with $z_1, z_2, z_3 \in \mathbb{R}^+$.

Then, $\sigma_3 \equiv 0$.

Under the assumption that $\mu_1(t; z) = \kappa(z_2 - z_1)$ is a mean-reverting term, we further find that $0 \le \mu_2(t; z) \le \frac{1}{2} z_3$ for all z, so that $\sigma_1(t; z) = \sqrt{z_3 - \mu_2(t; z)}$, $\sigma_2(t; z) = \sigma_1(t; z) - \sqrt{z_3 - 2\mu_2(t; z)}$ and $\mu_3(t; z) = -\mu_2(t; z)$.

In the case $\mu_2 = 0$, this yields $\sigma_2 = 0$ and then $\sigma_1(t; z) = \sqrt{z_3}$, which is the exponential Ornstein-Uhlenbeck stochastic volatility model discussed in depth by Fouque et al in [15].

3.3 Black & Scholes

The most trivial example of a variance curve functional is a constant function, i.e. $\mathcal{Z} = \mathbb{R}^+$ and

$$G(z; x) := z_1 \ . \tag{24}$$

This is the Black & Scholes variance curve in the sense that the prices of variance swaps are linear.

To satisfy (13) it is clear that $\mu_1(t; z_1)$ must vanish. The volatility function $\sigma_1(t; z_1)$ must be chosen such that $Z_t^1 \ge 0$. In other words, Z_t^1 must be a positive martingale. An example is the log-normal model, $Z_t := Z_0 \, \mathcal{E}(\int_0^t w(s) \, dW_s^1)$, where $\sigma_1(t; z) = z_1 w(t)$.

The next idea would be to set G to a fixed function,

$$G(z; x) := \gamma(x) \ .$$

However, it follows from (13) that this implies $\gamma(x) \equiv \alpha > 0$ is constant.

In reality, we will always employ some term-structure interpolation through observed implied volatilities in order to obtain a time-dependent Black & Scholes volatility function. A natural question is if we can use such methods to model the variance curve. It becomes even more important if we take into account the problem of recalibration in section 3.4 below.

3.3.1 Popular Volatility Term-structure Interpolation methods

In several texts, it is supposed to interpolate *implied volatility* by a "square root in time" or a logarithmic rule $\ln(1+x)$ (the latter has for example been proposed by Haffner p. 87 in [17]). The implied volatility at maturity x is defined as

$$\Sigma(z;x) = z_1 + z_2 w(x)$$

where w is for example

$$\begin{aligned} w_1(x) &:= \ln(1+x) && (25)\\ w_2(x) &:= \sqrt{\epsilon + x} && (26)\\ w_3(x) &:= 1/\sqrt{\epsilon + x} && (27) \end{aligned}$$

Hence, we may well test such an interpolation scheme proposed for a volatility surface to be used for a variance curve. In particular, if $\Sigma(z;x)$ is the implied volatility interpolation for maturity x with some parameter $z \in \mathcal{Z}$, then $\Sigma(z;x)^2 x$ is the overall implied variance, i.e. the price of a variance swap with maturity x.

In our previous notation,

$$G(z;x) := \partial_x \left(\Sigma(z;x)^2 x\right) = \Sigma(z;x)^2 + 2\Sigma(z;x)\Sigma'(z;x)x$$

in case $\Sigma(z;x) = z_1 + z_2 w(x)$, that gives

$$\begin{aligned} G(z;x) &= (z_1^2 + 2z_1 z_2 w(x) + z_2^2 w^2(x)) + 2(z_1 + z_2 w(x)) z_2 w'(x) x \\ &= z_1^2 + 2z_1 z_2 (w(x) + w'(x)x) + z_2^2 (w^2(x) + 2w(x)w'(x)x) \quad (28) \end{aligned}$$

We have to show:

$$\partial_x G \overset{!}{=} \mu(t;z)\,\partial_z G + \frac{1}{2}\,\sigma^2(t;z)\,\partial_{zz} G \quad (29)$$

We obtain

$$\begin{aligned} \partial_x G &= 2z_1 z_2 \{2w'(x) + w''(x)x\} \\ &\quad + 2z_2^2 \left\{2w'(x)w(x) + w'(x)^2 x + w''(x)w(x)x\right\} \\ \partial_{z_1} G &= 2z_1 + 2z_2 \{w(x) + w'(x)x\} \\ \partial_{z_2} G &= 2z_1 \{w(x) + w'(x)x\} + 2z_2 \left\{w^2(x) + 2w(x)w'(x)x\right\} \\ \partial_{z_1 z_1} G &= 2 \\ \partial_{z_2 z_2} G &= 2\left\{w^2(x) + 2w(x)w'(x)x\right\} \\ \partial_{z_1 z_2} G &= 2\{w(x) + w'(x)x\}\ . \end{aligned}$$

We now show for w_1 that $z_2 = 0$ and $\mu_1 = 0$, i.e. that the functional degenerates to the pure Black & Scholes case (24) if it is consistent. First, we compute the derivatives

$$w_1(x) = \ln(1+x)\ , \quad w_1'(x) = \frac{1}{1+x} \quad \text{and} \quad w_1''(x) = -\frac{1}{(1+x)^2}\ .$$

We therefore have

$$\partial_x G = 2z_1 z_2 \left\{ \frac{2}{1+x} - \frac{x}{(1+x)^2} \right\} + 2z_2^2 \left\{ 2\,\frac{\ln(1+x)}{1+x} + \frac{x - x\ln(1+x)}{(1+x)^2} \right\} .$$

Now note that none of the terms $\partial_{z_i} G$ or $\partial^2_{z_i z_j}$ contains terms in $\frac{1}{(1+x)^2}$. Hence, to satisfy (29), for all $x \geq 0$, we must have

$$0 \stackrel{!}{=} -2z_1 z_2 x + 2z_2^2 (x - x\ln(1+x)) . \tag{30}$$

Assume $z_2 \neq 0$. Then, (30) implies $z_1 = z_2(1 - \ln(1+x))$, which is not possible. Hence $z_2 = 0$ and the curve G reduces to the Black & Scholes case.
A similar computation for w_2 and w_3 shows the same result, i.e.

CONCLUSION 3.1 *None of the proposed interpolation schemes (25)-(27), if applied to variance interpolation, yields a consistent variance curve model except if reduced to the constant Black & Scholes case (24).*

3.3.2 Link to Stochastic Implied Volatility

Now consider the stochastic implied volatility model as proposed by Haffner [17] chapter 6 p. 116 equations (6.2)-(6.6). At any time t, the implied volatility for time-to-maturity x and relative log-strike k in his model is given by

$$\begin{aligned}\Sigma(z; x, k) \quad := \quad & z_1 + z_2 \ln(1+x) + \\ & (z_3 k\,(1 + \varrho_3 \ln(1+x)) + z_4 k^2\,(1 + \varrho_4 \ln(1+x))\end{aligned}$$

where $\varrho_3, \varrho_4 \in \mathbb{R}$ are constants. Hence, if Z is a consistent parameter process, the implied volatility at time t given a spot of S_t and parameters Z_t for cash strike K and maturity $T = t + x$ is given by $\Sigma\,(Z_t; T - t, \ln K/S_t)$.

With the results of the previous section, we then see that this model should not degenerate to the case $z_3 = z_4 = 0$, because then we obtain $\Sigma(\tilde{z}; x, k) = \tilde{z}_1 + \tilde{z}_2 \ln(1+x)$ which is only consistent if $\tilde{z}_2 = 0$.

In general, since every stochastic implied volatility model is naturally also a variance curve model, we note that we can use theorem 2.1 as a necessary condition to ensure that the resulting variance swap processes are martingales.

3.4 Recalibration of Stochastic Volatility Models

Another important consequence of the observations in section 3.1 is that daily recalibration of pricing models is subject to the same arbitrage conditions as developed above:

We will assume that an institution had *pricing model* $\mathbb{M}$ in place, which is parameterized by some parameter $z \in \mathcal{Z}$ and which then allows to compute prices of arbitrary claims on S.

As a guiding example, we will use Heston's model: Let $\tilde{W} = (\tilde{\Omega}, \tilde{\mathcal{F}}_\infty, \tilde{\mathbb{P}}, \tilde{\mathbb{F}})$ be the standard Wiener space with two-dimensional Brownian motion $\tilde{W} = (\tilde{W}^1, \tilde{W}^2)$. Then define

$$d\tilde{v}_\tau := \kappa(m - \tilde{v}_\tau)\, dt + \nu\sqrt{\tilde{v}_\tau}\, d\tilde{W}^1_\tau \qquad \tilde{v}_0 = v_0 \tag{31}$$

$$d\tilde{X}_\tau := \sqrt{\tilde{v}_\tau}\, d(\rho\tilde{W}^1_\tau + \sqrt{1-\rho^2}\tilde{W}^2_\tau) \qquad \tilde{X}_0 = 0 \tag{32}$$

with $z = (S_0, v_0; \kappa, m, \nu, \rho) \in \mathcal{Z} = \mathbb{R}^{+4} \times (-1, +1)$ such that $\tilde{S} = S_0\, \mathcal{E}(\tilde{X})$ is a martingale.

We now consider this as a model with parameters $z \in \mathcal{Z}$. It is clear that *inside the model* in *model time* τ only the states S_τ and v_τ change while the remaining four parameter are assumed to be constant. (We have used the time-index τ to underline the fact that τ denotes *model time*.)

On this space, we denote by θ the shift-operator as in Revuz/Yor [25] p. 35. Let H be a payoff dependent on the path of $\tilde{S}$, i.e. a measurable and integrable map

$$H : C[0, \infty) \to \mathbb{R} \ .$$

Given a continuous non-random function $(Y_u)_{u\in[0,t]}$ with $Y_t = \tilde{S}_0$ we can nonetheless define

$$H \circ \theta^Y_t(\tilde{S}) := H(Y 1_{[0,t)} + \tilde{S} \circ \theta_{-t} 1_{[t,\infty)}) \ .$$

This formalizes the idea of "gluing" Y in front of $\tilde{S}$ to account for the unavailable information before $\tau = 0$.

Now assume that we are trading in a market with continuous stock price process S and a constant cash bond of 1. Besides the stock S, we assume that at any time t, prices of various reference instruments such as European option prices are publicly quoted. We use their prices $P_t = (P^i_t)_{i=1,\dots,N_t}$ to *calibrate* our pricing model, i.e. we run an algorithm

$$\Psi : P_t \mapsto Z_t \in \mathcal{Z}$$

which uniquely determines some model parameters Z_t. We use these parameters to value our exotic option positions (the term "exotic" here means that there is no publicly available market price for the product). We denote by $\mathcal{H}$ the set of payoffs of those exotic structures. At any time t the price of such an exotic option $H \in \mathcal{H}$ is then given by

$$\pi_t[H] := \tilde{\mathbb{E}}^{Z_t}\left[\, H \circ \theta^S_t(\tilde{S}) \,\right]$$

(note that past fixings etc are encoded in the θ^S shift). It is apparent that this procedure yields a new *meta-model* $\mathcal{M}$ by means of successive recalibration which is given by the price processes of the exotics in the book of the institution and the stock price itself

$$\mathcal{M} = (S; \pi(H)_{H\in\mathcal{H}}) \ .$$

Clearly, there are two categories of arbitrage in such a meta-model: The first, which we want to call *static arbitrage*, is arbitrage within the underlying pricing model $\mathbb{M}$ itself (such as violations of European call price relations etc). The second, we call it *dynamic arbitrage*, relates to the meta-model:

Under which conditions do the price processes in $\mathcal{M}$ generate arbitrage?

In the light of our variance curve models, it is clear that the implied variance curve functional of the meta-model must be consistent. This is a necessary condition for any meta-model and can therefore be applied under the assumption of continuity of the price process S and the parameter process Z. It follows:

PROPOSITION 3.1 *The Heston meta-model is not free of dynamic arbitrage if the speed κ of mean-reversion in Z_t is not kept constant.*

PROPOSITION 3.2 *An meta-model which uses Black & Scholes' pricing model with one of the term-structure interpolation of section 3.3.1 admits dynamic arbitrage if the initially calibrated term-structure is recalibrated.*

Given the findings above, a more convenient term-structure interpolation for implied volatility scheme is given by the "mean-reverting" curves introduced above. These curves can be recalibrated more safely (if the exponential parameters are kept constant) and provide a good fit to real data.

4 Conclusions

We have developed a framework for variance curve market models, both in infinite dimensions and in finite-dimensional representations. We have derived necessary and sufficient conditions to ensure that a variance curve functional is consistent with some parameter process and applied the results to various examples which are used in practise. We have shown that mean-reversion in the analyzed models must be kept constant and that various implied-volatility term-structure functionals must be used with great care before used in daily recalibration. A specific variance curve functional, namely the "Double mean-reverting model", has been proposed as an application.

Further directions of research can be the introduction of jumps on one hand side, and an application of more recent results to absence of arbitrage in infinite dimensions.

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