Let's Get Real

Wrap up and Computations

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1. Express the *regular* hexagon as a basic semialgebraic set over $\mathbb{Q}$. Is that hexagon a spectrahedron over $\mathbb{Q}$? A spectrahedral shadow over $\mathbb{Q}$?

2. Give an explicit homogeneous polynomial $f \in \mathbb{Z}[x_0, x_1, x_2, x_3]$ of degree 3 whose *cubic surface* in $\mathbb{P}^3$ is smooth and contains precisely 15 real lines. Can you arrange for the coefficients of f to lie in $\{-2, -1, 0, 1, 2\}$?

3. What is the maximum number of connected components of a curve in $\mathbb{RP}^3$ that is the intersection of a quadric and a cubic? Give an example.

4. What is the maximum number of connected components of a surface in $\mathbb{RP}^4$ that is the intersection of a quadric and a cubic? Give an example.

5. How many real inflection points can a plane cubic curve have? Prove your claim. Express the coordinates of the inflection points in radicals.

6. For 1000 random polynomials $f(x, y)$ of degree 5 with real coefficients, compute the global minimum of $x^6 + y^6 + f(x, y)$ in $\mathbb{R}^2$. How many real critical points are there on average? How many local minima?

7. Consider the principal ideal in $\mathbb{R}[x_1, x_2, x_3, x_4, x_5]$ that is generated by $\sum_{i=1}^{5} \prod_{j \in \{1,\ldots,5\} \setminus \{i\}} (x_i - x_j)$. Determine the *real radical* of this ideal, i.e. the vanishing ideal of the set of all real zeros of this polynomial.

8. Arrange for your cubic surface $V(f)$ in # 2 to contain $p = (0:0:0:1)$, and project $V(f)$ onto $\mathbb{P}^2$ from center p. Draw the branch curve in $\mathbb{RP}^2$, and also draw the images of the 15 lines. What happens to p in $\mathbb{P}^2$?

9. Identify the topology of each smooth quartic curve in $\mathbb{RP}^2$ in the pencil

$$25(x^4+y^4+z^4)-34(x^2y^2+x^2z^2+y^2z^2)+t\cdot\big((x-z)^4+(x-y+z)^4+(y+2z)^4\big)$$

As t ranges over $\mathbb{R}$, which of the six types occur, and how often?

10. Pick an explicit hyperbolic polynomial of degree 6 in $\mathbb{R}[x_0, x_1, x_2]$ and draw the hyperbolic curve. Can you find an example where the outer oval is convex but the middle oval is not? How about in $\mathbb{R}[x_0, x_1, x_2, x_3]$?

11. For the previous problem, sketch the Riemann surface, indicate the involution given by complex conjugation, and highlight the real curve.

12. Give an explicit cubic spectrahedron in $\mathbb{R}^3$ that has only two real nodes.

13. Arrange for your curve in # 3 to be compact in $\mathbb{R}^3$. Describe all faces of its convex hull. Express this convex body as a spectrahedral shadow.

14. Consider the preordering of all polynomials in two variables that are nonnegative on a regular hexagon. Is it finitely generated? Stable? Archimedian? Does the choice of field (like $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{R}$ or $\mathbb{R}\{\{\epsilon\}\}$) matter?

15. Repeat the first task in # 6 using sum of squares (SOS) optimization, where you maximize λ such that $x^6 + y^6 + f(x, y) - \lambda$ is a SOS.

16. Compare random real curves of degree 4 in $\mathbb{P}^2$ with random real curves of degree $(2, 5)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Which has more connected components?

17. True or false: every nonnegative ternary quartic over $\mathbb{R}$ is a sum of 4th powers of linear forms over $\mathbb{R}$. How to compute such a representation?

18. True or false: every combinatorial type of 3-polytope with 8 vertices arise as the convex hull of the intersection of three real quadrics in $\mathbb{R}^3$. What if we restrict to simplicial polytopes?

19. Describe the Torelli map for complex K3 surfaces as explicitly as possible. Which period points correspond to quartic surfaces in $\mathbb{P}^3$? Real quartic surfaces in $\mathbb{P}^3$? Real quartic surfaces in $\mathbb{P}^3$ with no real points?

20. Pick one of the 35 tropical quadrilaterals in Figure 5.2.4 of my book with Diane Maclagan. Determine four points in the positive orthant of $\mathbb{R}\{\{\epsilon\}\}^2$ whose convex hull exhibits your choice as its tropicalization.