Exercise Session on Random Real Algebraic Geometry

Exercise 1. Let $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ be a random matrix where the $a_{i,j}$ are all independent random variables with $a_{i,j} \sim N(0,1)$ for all i,j. Prove that

(1) $\mathbb{E}\det(A) = 0$.
(2) $\mathbb{E}\det(A)^2 = n!$.

Suppose now that $a_{i,j} \sim N(0,\sigma_{i,j}^2)$ and let $S = (\sigma_{i,j}^2)$ denote the matrix having the $\sigma_{i,j}^2$ as entries. Show

(3) $\mathbb{E}\det(A)^2 = \mathrm{per}(S)$.

Exercise 2. Let $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ be a random matrix with $a_{1,1},\ldots,a_{n,n} \overset{\text{iid}}{\sim} N(0,1)$. Use Vitale's theorem (Theorem 1 below) to show that

$$\mathbb{E}\,|\det(A)| = \frac{n!}{\sqrt{2}^n\,\Gamma\left(\frac{n}{2}+1\right)}.$$

Hint: The volume of an n-dimensional ball[1] with radius r is $\sqrt{\pi}^n r^n \Gamma(\frac{n}{2}+1)^{-1}$.

Exercise 3 (Probabilistic Bézout-Theorem)**.** Put $\mathcal{R}_d := \mathbb{R}[x_0,\ldots,x_n]_d$. Let us write $f \sim G(d)$, if $f \in \mathcal{R}_d$ is the following random homogeneous polynomials of degree d:

$$f = \sum_{\alpha_0+\ldots+\alpha_n=d} \lambda_{(\alpha_0,\ldots,\alpha_n)} \sqrt{\tfrac{d!}{\alpha_0!\cdots\alpha_n!}}\; x_1^{\alpha_0}\cdots x_n^{\alpha_n}; \qquad \lambda_{(\alpha_0,\ldots,\alpha_n)} \overset{\text{iid}}{\sim} N(0,1).$$

One says that f is standard Gaussian w.r.t. the Bombieri norm $\|f\| := (\sum_{\alpha_0+\ldots+\alpha_n=d} (\lambda_{(\alpha_0,\ldots,\alpha_n)})^2)^{\frac{1}{2}}$. The orthogonal group $\mathcal{O}(n)$ acts on $\mathcal{R}_d$ via $U.f := f\circ U^{-1}$. The Bombieri norm has the property that $\|U.f\| = \|f\|$ for all U and f.
Let $\mathbf{d} := (d_1,\ldots,d_n)$ be an n-tuple of degrees and define the random ideal

$$I_{\mathbf{d}} := (f_1,\ldots,f_n), \qquad f_i \sim G(d_i), \text{ all } f_i \text{ are independent.}$$

In this exercise we show that $E := \mathbb{E}[\,\#V(I_{\mathbf{d}})\,] = \sqrt{d_1\cdots d_n}$ (here $V(\cdot)$ means the zero set in $\mathbb{RP}^n$).

(1) Write down the joint density of the coefficients of the f_i in terms of the Bombieri norm. Conclude that the density $\varphi(f)$ of $f = (f_1,\ldots,f_n)$ is invariant under coordinate change.
(2) Interpret f as a function $\mathbb{S}(\mathbb{R}^{n+1}) \to \mathbb{R}^n$. Use the Kac-Rice formula (Theorem 2 below) to write E as an integral over $\mathbb{S}(\mathbb{R}^{n+1})$.
(3) Step (2) yields the nested integral $E = \frac{1}{2}\int_{\mathbb{S}(\mathbb{R}^{n+1})} \left[\int_{\{f\mid f(x)=0\}} |\det f'(x)|\;\varphi(f)\;\mathrm{d}f\right]\,\mathrm{d}x$. Use the orthogonal invariance of the Bombieri norm to show that the inner integral is independent of x.
(4) Let $e := (1,0,\ldots,0)$. Show that

$$f'(e) = \begin{pmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & & \vdots \\ f_{n,1} & \cdots & f_{n,n} \end{pmatrix},$$

where $f_{i,j}$ = coefficient of $x_0^{d_i-1}x_j$ in f_i.
Hint: The tangent space of $\mathbb{S}(\mathbb{R}^{n+1})$ at e is $e^\perp = \left\{x\in\mathbb{R}^{n+1} \mid e^T x = 0\right\}$.
(5) Conclude that

$$E = \frac{\mathrm{vol}(\mathbb{S}(\mathbb{R}^{n+1}))}{2\sqrt{2\pi}^n}\;\mathop{\mathbb{E}}_{A}|\det(\mathrm{diag}(\sqrt{d_1},\ldots,\sqrt{d_n})\,A)|,$$

where $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ is a random matrix with $a_{1,1},\ldots,a_{n,n} \overset{\text{iid}}{\sim} N(0,1)$.
(6) Use **Exercise 2** to compute E.
Hint: The volume of $\mathbb{S}(\mathbb{R}^{n+1})$ is $2^{n+1}\sqrt{\pi}^n\Gamma(\frac{n}{2}+1)(n!)^{-1}$.

[1] See http://dlmf.nist.gov/5.19#iii

Exercise 4. The projective spaces $\mathbb{RP}^n, \mathbb{CP}^n$ are given a riemannian structure for which the canonical projections $\mathbb{S}^n \to \mathbb{RP}^n, \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ are riemannian submersions. Here $\mathbb{S}^n$ is endowed with the standard metric. Given a real degree d hypersurface $X \subset \mathbb{CP}^n$ we denote by $|X|$ and $|X_{\mathbb{R}}|$ the volume of X and its real part $X_{\mathbb{R}} \subset \mathbb{RP}^n$ respectively. Prove that

$$\frac{|X_{\mathbb{R}}|}{|\mathbb{RP}^{n-1}|} \leq \frac{|X|}{|\mathbb{CP}^{n-1}|}$$

Hint: Use Theorem 3.

Exercise 5. Prove that for a random degree d hypersurface $X = \{f = 0\} \subset \mathbb{RP}^n$ defined by $f \sim G(d)$ ($G(d)$ as in Exercise 3) the following formula holds

$$\mathbb{E}\,|X| = \sqrt{d}\,|\mathbb{RP}^{n-1}|$$

Hint: Use Theorem 3 and orthogonal invariance of the distribution.

Exercise 6. (Expected number of critical rank-one approximations to a random symmetric tensor) In [3, Theorem 1.5] we find the following statement: When a symmetric tensor $v \in (\mathbb{R}^n)^{\otimes p}$ is standard Gaussian w.r.t. the Bombieri norm, the expected number of critical rank-one approximations to v is

$$C(n) \int\limits_{\lambda_1 \leq \ldots \leq \lambda_{n-1}} \int\limits_{w \in \mathbb{R}} \prod_{i=1}^{n-1} \left| \sqrt{\tfrac{p}{2}}\, w - \sqrt{p-1}\, \lambda_i \right| \prod_{i<j} (\lambda_j - \lambda_i)\ \exp\left(-\tfrac{w^2}{4} - \sum_{i=1}^{n-1} \tfrac{-\lambda_i^2}{4} \right) \mathrm{d}w \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_{n-1}.$$

where $C(n)^{-1} = 2^{(n^2+3n)/4} \prod_{i=1}^n \Gamma\left(\frac{i}{2}\right)$. Show that for $p = 2$ this integral evaluates to n. There is an argument why for $p = 2$ this integral necessarily must equal n. Can you find it?
Hint: This integral looks pretty scary. In fact, evaluating the integral directly is hard, so we don't want to do that. Instead, we will employ symmetries of the integrand. Show that for $p = 2$ the integrand is invariant under permutations of the variables $\lambda_1, \ldots, \lambda_{n-1}, w$ and deduce that you can write the integral above in the form $\int_{\lambda_1 \leq \ldots \leq \lambda_n} \prod_{i<j} (\lambda_j - \lambda_i)\ \exp(-\sum_{i=1}^n \frac{-\lambda_i^2}{4})\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n$. Then, use the following corollary of [4, Theorem 3.2.17], that says $C(n) \int_{\lambda_1 \leq \ldots \leq \lambda_n} \prod_{i<j} (\lambda_j - \lambda_i)\ \exp(-\sum_{i=1}^n \frac{-\lambda_i^2}{4})\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n = 1$.

The following two exercises are not concerned with randomness, but instead are meant to give a better understanding of critical rank-one approximations of tensors.

Exercise 7. Let $e_1 = (1, 0), e_2 = (0, 1) \in \mathbb{R}^2$ and $d_1 = (1, 0, 0), d_2 = (0, 1, 0) \in \mathbb{R}^2$. Compute the real critical rank-one approximations of the tensor $T = e_1 \otimes d_1 \otimes d_1 + e_2 \otimes d_2 \otimes d_2 \in \mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$.

Exercise 8. Let e_1, e_2, e_3 be the three standard basis vectors in $\mathbb{R}^3$. Compute all the real symmetric rank-one approximations of the following symmetric tensors.

(1) $T_1 = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} \in S^3(\mathbb{R}^3)$.
(2) $T_2 = e_1^{\otimes 4} + e_2^{\otimes 4} + e_3^{\otimes 4} \in S^4(\mathbb{R}^3)$.

Hint: It can be useful to consider the tensors as polynomials, because critical rank-one approximations correspond to critical points on the unit sphere of those polynomials. Can you show why?

References

[1] Robert J. Adler and Jonathan E. Taylor. *Random fields and geometry.* Springer, 2007.
[2] Peter Bürgisser and Felipe Cucker. *Condition*, volume 349 of *Grundlehren der Math. Wiss.* Springer, Heidelberg, 2013.
[3] J. Draisma and E. Horobet. The average number of critical rank-one approximations to a tensor. *Linear and Multilinear Algebra*, 64(12):2498–2518, 2016.
[4] R.J. Muirhead. *Aspects of Multivariate Statistical Theory*, volume 131. John Wiley & Sons, NY, 1982.
[5] Richard A. Vitale. Expected absolute random determinants and zonoids. *Ann. Appl. Probab.*, 1(2):293–300, 05 1991.

Supplementary Material

A *standard Gaussian random variable* x has the density $\varphi(x) = \sqrt{2\pi}^{-1} \exp(-\frac{1}{2} x^2)$. One also writes $x \sim N(0,1)$ (N for 'normal distribution'). Consequently, the random vector $x = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n \overset{\text{iid}}{\sim} N(0,1)$ has the density $\prod_{i=1}^n \varphi(x_i) = \sqrt{2\pi}^{-n} \exp(-\frac{1}{2} x^T x)$.

Theorem 1 (Vitale's theorem [5, Theorem 3.2])**.** *Let $Y \in \mathbb{R}^n$ be a random vector and let $M_Y \in \mathbb{R}^{n\times n}$ be a random matrix whose columns are iid copies of Y. Consider the function*

$$h_Y(x) := \mathop{\mathbb{E}}_{Y} \max\left\{\langle x, tY\rangle \mid 0 \le t \le 1\right\}.$$

The function h_Y is a support function, to which is associated some convex body $B \subset \mathbb{R}^n$. Then, we have

$$\mathbb{E}|\det(M_Y)| = n!\,\mathrm{vol}(B)$$

(in fact, B is a zonoid, and it is called the expected zonoid of Y).

Theorem 2 (Kac-Rice formula for Gaussian polynomial functions [1, Theorem 12.1])**.** *Let M be a compact oriented, n-dimensional C^1 manifold with a C^1 Riemannian metric g (for instance the unit sphere). Let*

$$f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$$

be random polynomials, whose coefficients are Gaussian random variables und let $u \in \mathbb{R}^n$. We denote by $\varphi(f)$ the density of f and put

$$N_u := \#\left\{x \in M \mid f(x) = u\right\}$$

Then

$$\mathop{\mathbb{E}}_{f} N_u = \int_M \left[\int_{\{f \mid f(x)=u\}} |\det f'(x)|\, \varphi(f)\, \mathrm{d}f\right] \mathrm{d}x.$$

Remark. Theorem 2 holds in much more generality, where the multivariate function f can have any distribution that satisfies certain variance constraints and whose density p is sufficiently continuous.

Theorem 3 (A corollary of the Poincaré formula [2, Theorem 20.9])**.** *Let $O(n+1)$ and $U(n+1)$ denote the orthogonal and unitary group, respectively. The following holds.*

(1) Let $V, W \subset \mathbb{RP}^n$ be smooth irreducible projective varieties of dimensions m and $n-m$. Then, we have

$$\mathop{\mathbb{E}}_{u\in O(n+1)} \#(V \cap uW) = \frac{|V|}{|\mathbb{RP}^m|}\,\frac{|W|}{|\mathbb{RP}^{n-m}|}$$

(2) Let $V, W \subset \mathbb{CP}^n$ be smooth irreducible projective varieties of dimensions m and $n-m$. Then, we have

$$\mathop{\mathbb{E}}_{u\in U(n+1)} \#(V \cap uW) = \frac{|V|}{|\mathbb{CP}^m|}\,\frac{|W|}{|\mathbb{CP}^{n-m}|}.$$

Here projective spaces $\mathbb{RP}^n, \mathbb{CP}^n$ are given a riemannian structure for which the canonical projections $\mathbb{S}^n \to \mathbb{RP}^n, \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ are riemannian submersions and $\mathbb{S}^k$ is given the standard metric.