Numerical computation of the homology of basic semialgebraic sets

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Inria Saclay

joint work with Peter Bürgisser and Felipe Cucker

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Linking Topology to Algebraic Geometry and Statistics

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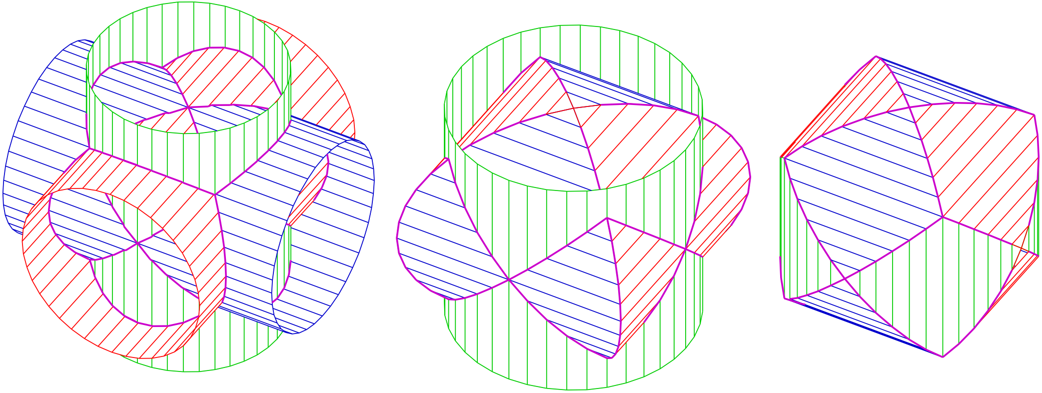
Basic semialgebraic sets



definition A *basic semi algebraic set* is the solution set of finitely many polynomial equation and inequations.

Picture : `https://de.wikipedia.org/wiki/Steinmetz-Körper`

J. T. Schwartz, M. Sharir, “On the *piano movers* problem”

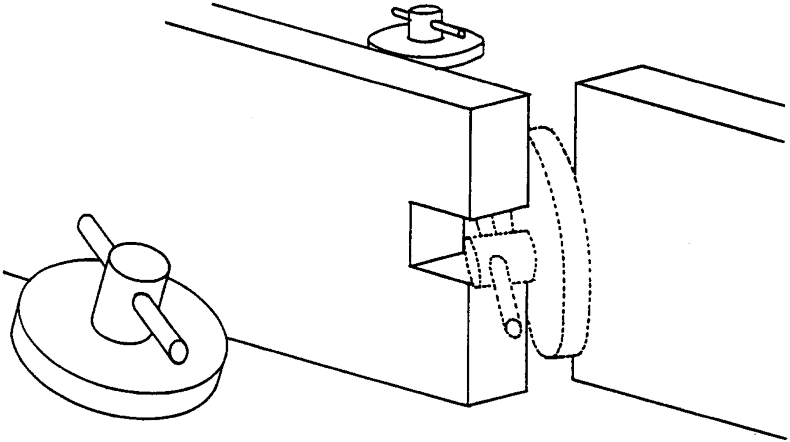


FIG. 1. An instance of our case of the “piano movers” problem. The positions drawn in full are the initial and final positions of B; the intermediate dotted positions describe a possible motion of B between the initial and final positions.

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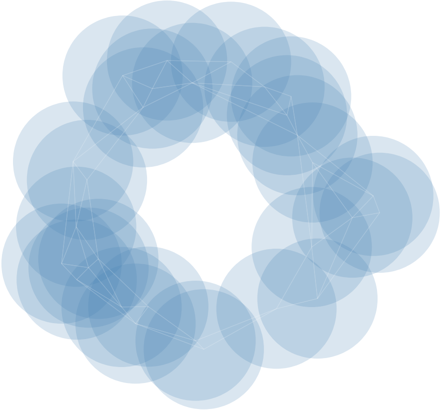
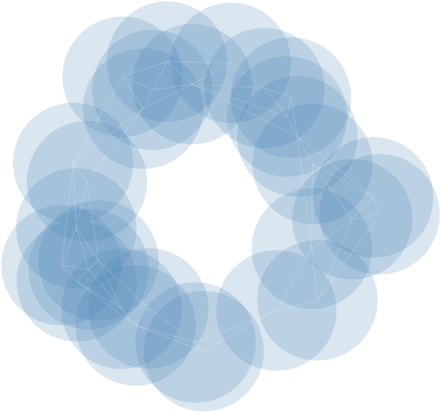
CAD Compute the cylindrical algebraic decompositon (Collins)



- Homotopically equivalent to its *nerve*



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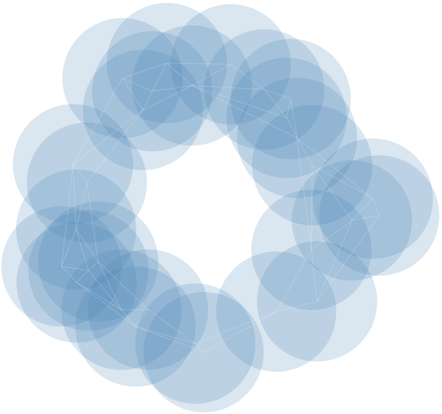
- Combinatorial computation of the homology



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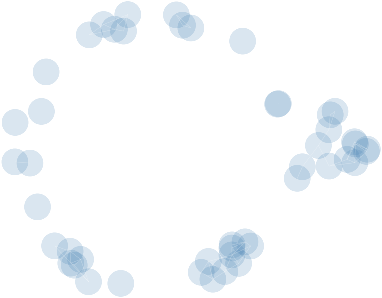
Approaching sets by union of balls



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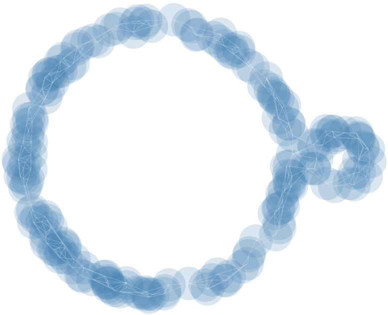
  - sufficiently many points



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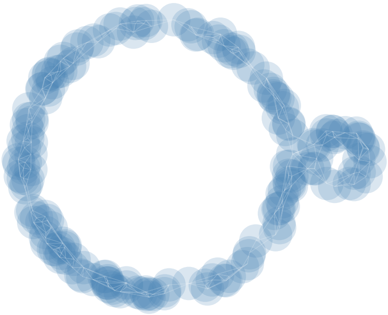

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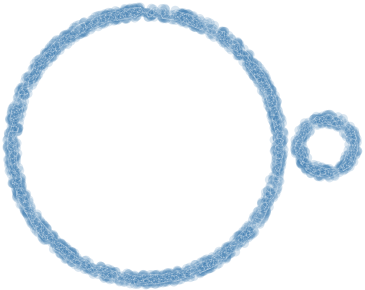

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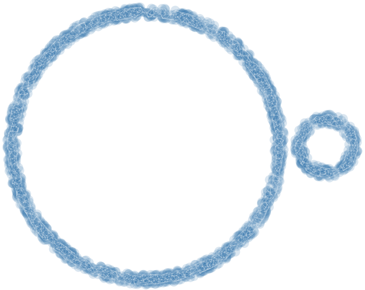

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- How to quantify “sufficiently many”, “too small” and “too large” in an algebraic setting?
- Can we derive algebraic complexity bounds for the computation of the homology of semialgebraic sets?

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probability measure	Gaussian probability distribution
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grid methods Initiated by Cucker, Krick, Malajovich, Shub, Smale, Wschebor

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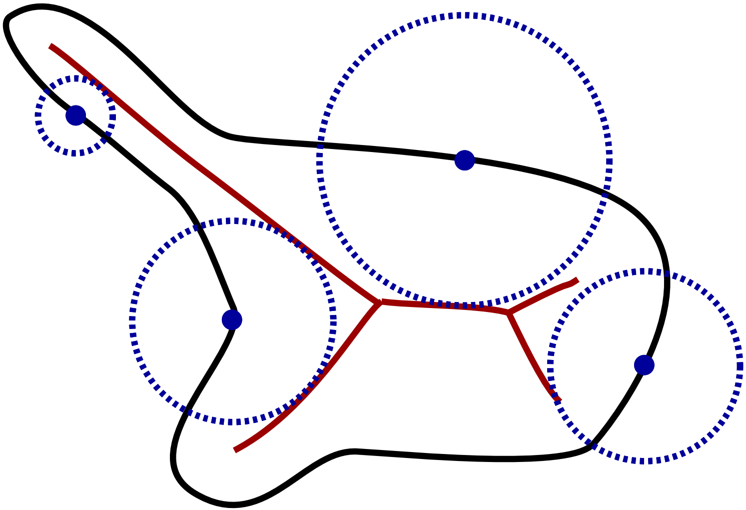
Is there a considition number for *closed sets*?

Reach of a closed set



The reach of a set is its minimal distance to its medial axis.

https://en.wikipedia.org/wiki/Local_feature_size

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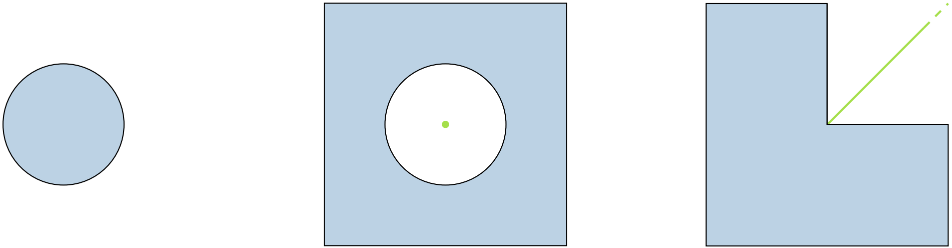

$\tau(W) = \infty$ $\quad$ $\infty > \tau(W) > 0$ $\quad$ $\tau(W) = 0$

The Niyogi-Smale-Weinberger theorem

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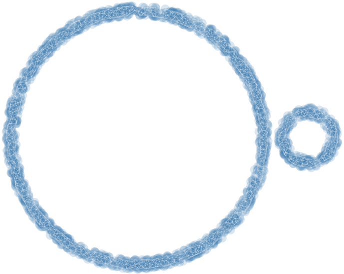
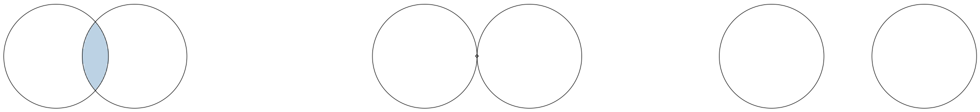
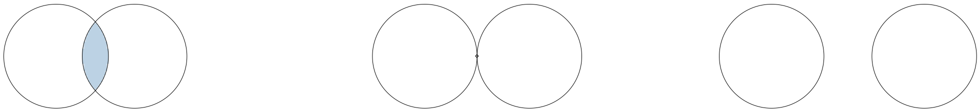




Non-transversal intersection of the boundaries



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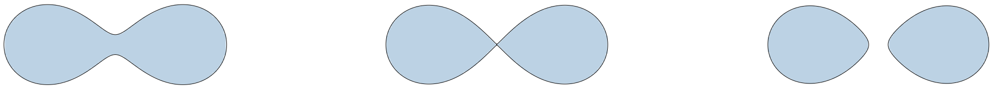

Singularity in the boundary

homogeneous setting $X \subset \mathbb{S}^n$ defined by homogeneous polynomial equations $f_1 = 0, \ldots, f_q = 0$ (denoted $F = 0$) of degree at most D.

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condition number $\kappa(F) = \|F\| / \operatorname{dist}(F, \{\text{ill-posed problems}\})$.

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example A cubic plane curve:

$$a_0 x^3 + a_1 x^2 + a_2 x y^2 + a_3 y^3 + a_4 x^2 + a_5 x y + a_6 y^2 + a_7 x + a_8 y + a_9 = 0.$$

$\dim \mathscr{H} = 9$ and the ill-posed set is given by the following degree 12 polynomial with 2040 monomials

$$-19683 a_0^4 a_3^4 a_9^4 + 26244 a_0^4 a_3^3 a_6 a_8 a_9^3 - 5832 a_0^4 a_3^3 a_8^3 a_9^2 - 5832 a_0^4 a_3^2 a_6^3 a_9^3 - 7290 a_0^4 a_3^2 a_6^2 a_8^2 a_9^2 + 38\dots$$
$$-1836 a_0^4 a_3 a_6^3 a_8^3 a_9 + 216 a_0^4 a_3 a_6^2 a_8^5 - 432 a_0^4 a_6^6 a_9^2 + 216 a_0^4 a_6^5 a_8^2 a_9 - 27 a_0^4 a_6^4 a_8^4 + 26244 a_0^3 a_1 a_2 a\dots$$
$$+3888 a_0^3 a_1 a_2 a_3 a_6^3 a_9^3 + 4860 a_0^3 a_1 a_2 a_3 a_6^2 a_8^2 a_9^2 - 2592 a_0^3 a_1 a_2 a_3 a_6 a_8^4 a_9 + 288 a_0^3 a_1 a_2 a_3 a_8^6 - 12\dots$$
$$-8748 a_0^3 a_1 a_3^3 a_5 a_8 a_9^3 - 8748 a_0^3 a_1 a_3^3 a_6 a_7 a_9^3 + 5832 a_0^3 a_1 a_3^3 a_7 a_8^2 a_9^2 + 5832 a_0^3 a_1 a_3^2 a_5 a_6^2 a_9^3 + 486\dots$$
$$+4860 a_0^3 a_1 a_3^2 a_6^2 a_7 a_8 a_9^2 - 5184 a_0^3 a_1 a_3^2 a_6 a_7 a_8^3 a_9 + 864 a_0^3 a_1 a_3^2 a_7 a_8^5 - 5184 a_0^3 a_1 a_3 a_5 a_6^3 a_8 a_9^2 + \dots$$
$$+1836 a_0^3 a_1 a_3 a_6^3 a_7 a_8^2 a_9 - 360 a_0^3 a_1 a_3 a_6^2 a_7 a_8^4 + 864 a_0^3 a_1 a_5 a_6^5 a_9^2 - 360 a_0^3 a_1 a_5 a_6^4 a_8^2 a_9 + 36 a_0^3 a_1\dots$$

Distance to ill-posedness

theorem $$\operatorname{dist}(F, \{\text{ill-posed}\}) \simeq \min_{x \in \mathbb{S}^n} \underbrace{\left(\frac{1}{\|\mathrm{d}_x F^\dagger\|^2} + \|F(x)\|^2 \right)^{\frac{1}{2}}}_{\text{vanisihes at a singular root}}$$

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$\rightsquigarrow$ $\kappa(F)$ is easily approximable.

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An algebraic condition number

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condition number $\kappa_*(F, G) = \max_{L \subseteq G} \kappa(F \cup L)$.

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condition number $\kappa_*(F, G) = \max_{L \subseteq G} \kappa(F \cup L)$.

theorem $\kappa_*(F, G) \leqslant \|F, G\| / \operatorname{dist}((F, G), \{\text{ ill-posed problems }\})$.

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corollary $\mathscr{X} \subset \mathbb{S}^n$ finite.

For any $\delta \in \left(3 \operatorname{dist}_{\text{Hausdorff}}(\mathscr{X}, W), \left(14 D^{\frac{3}{2}} \kappa_*(F, G)\right)^{-1}\right)$,

$$\bigcup_{x \in \mathscr{X}} B_\delta(x) \cong W.$$

Sampling and thickening

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efficiency How to check $\operatorname{dist}(x, W) \leqslant \frac{1}{3}\delta$?

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idea Replace $\mathrm{dist}(x, W) \leqslant \frac{1}{3}\delta$ by $x \in W(r)$ (for a suitable r).

Covering algorithm

input A spherical semialgebraic set $W = \left\{x \in \mathbb{S}^n \mid F(x) = 0, G(x) \geqslant 0\right\}$

assumption $\kappa_*(F, G)$ is finite.

output A finite set $\mathscr{X} \subset \mathbb{S}^n$ and an $\varepsilon > 0$ such that $B_\varepsilon(\mathscr{X}) \cong W$.

algorithm

function COVERING(F, G)
    $r \leftarrow 1$
    repeat
        $r \leftarrow r/2$
        Compute a r-grid $\mathscr{G}_r$ in $\mathbb{S}^n$
        $k_* \leftarrow \max\{\kappa(F \cup L, x) \mid x \in \mathscr{G}_r$ and $L \subseteq G\}$
    until $71\, D^{\frac{5}{2}} k_*^2 r < 1$
    return the set $\mathscr{X} = \mathscr{G}_r \cap W(D^{\frac{1}{2}} r)$ and the real number $\varepsilon = 5Dk_* r$
end function

Complexity analysis

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Weak complexity bounds

If the average case is unbounded, is the algorithm slow?

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example The power method for computing the dominant eigenpair of a real $d \times d$ symmetric matrix (compute $M^n x$ for large n).

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Used in practice with success.

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weak complexity cost $\leqslant \mathrm{poly}(d)$ with probability $\geqslant 1 - \exp(-d)$.

(Amelunxen, Lotz)

general bound If $\Sigma \subset \mathscr{H}$ is an homogeneous algebraic hypersurface, and if $X \in \mathscr{H}$ is a Gaussian isotropic random variable,

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corollary 2 cost $\leqslant 2^{O(N^2)}$ with probabiliy $\geqslant 1-2^{-N}$.

Perspectives

Ill-posedness is relative to a data representation

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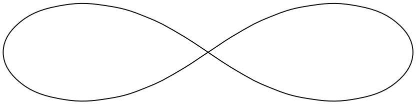
example Given by a rational parametrization, the lemniscate is well-conditionned



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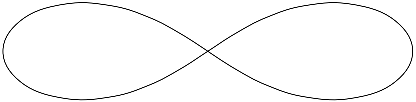

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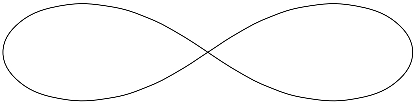

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Thank you!