Persistent Homology and the Stability Theorem

Ulrich Bauer

TUM

February 19, 2018

TAGS – Linking Topology to Algebraic Geometry and Statistics

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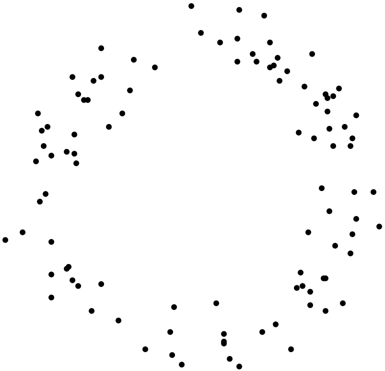

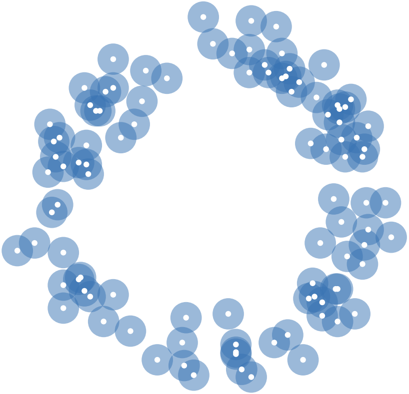

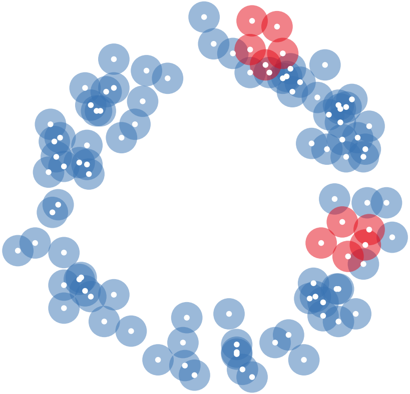

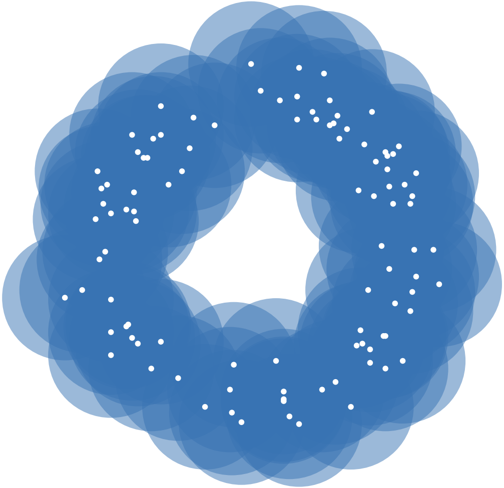

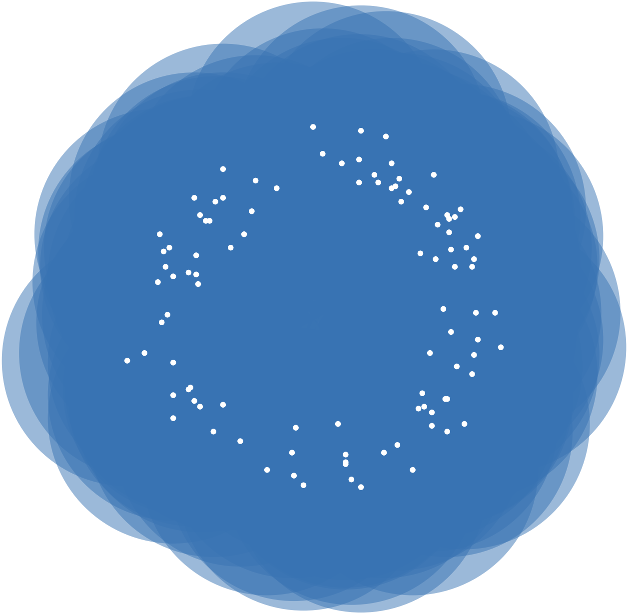

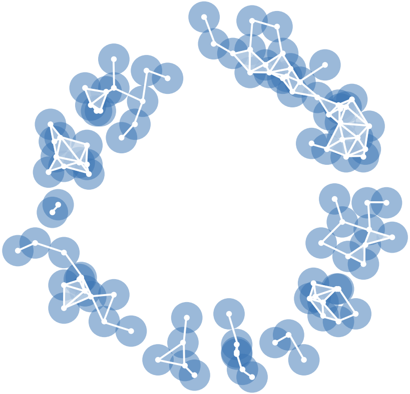

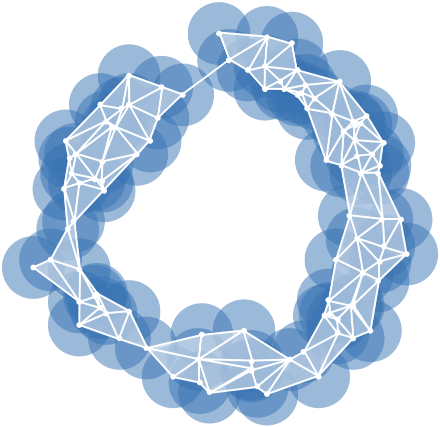

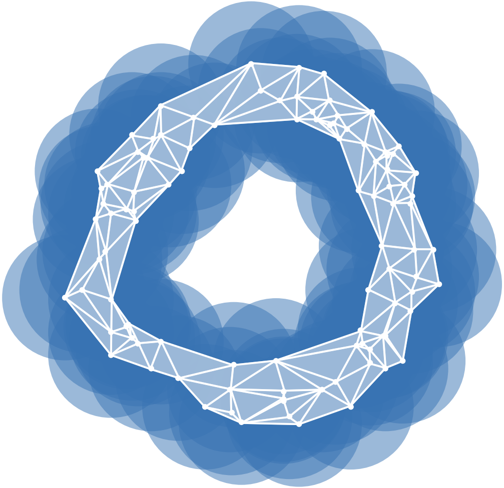

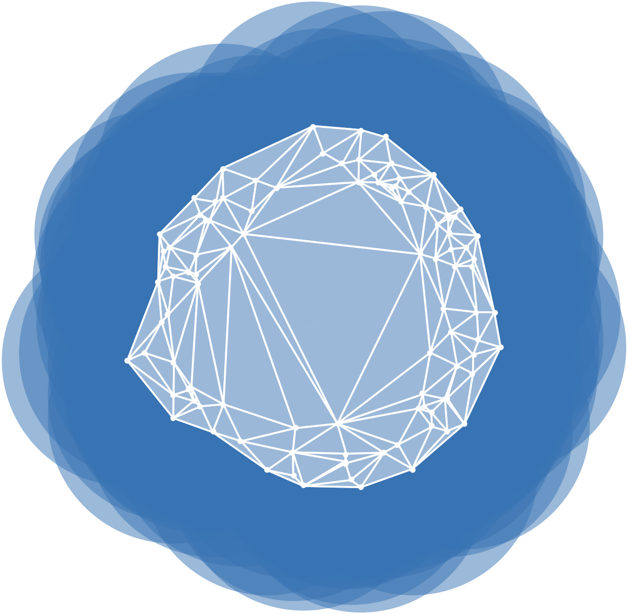

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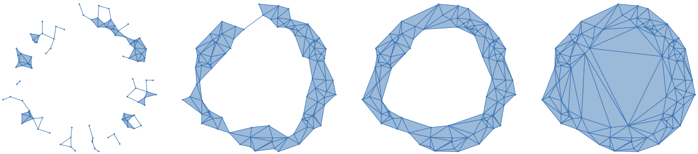

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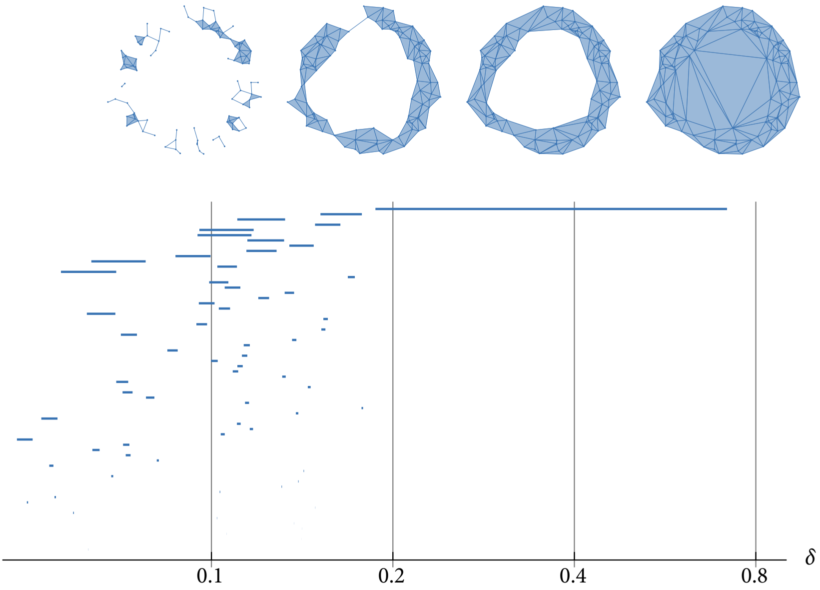

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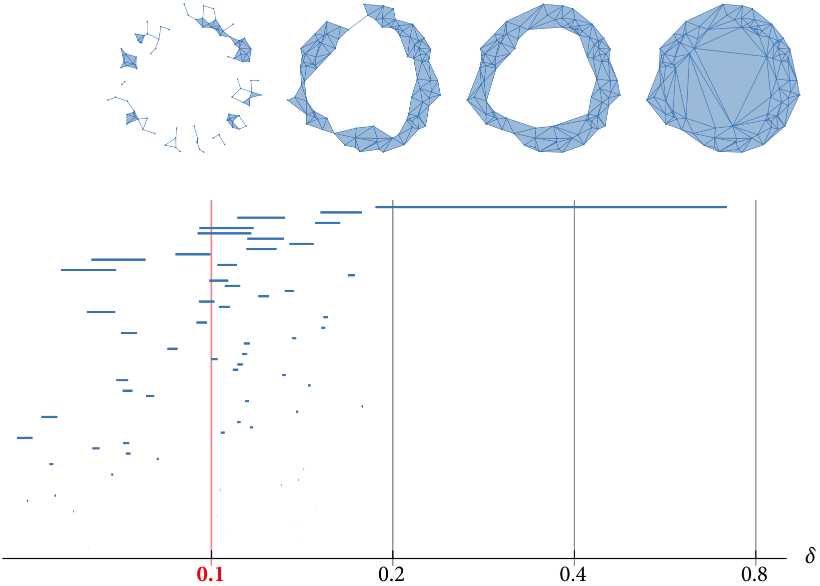

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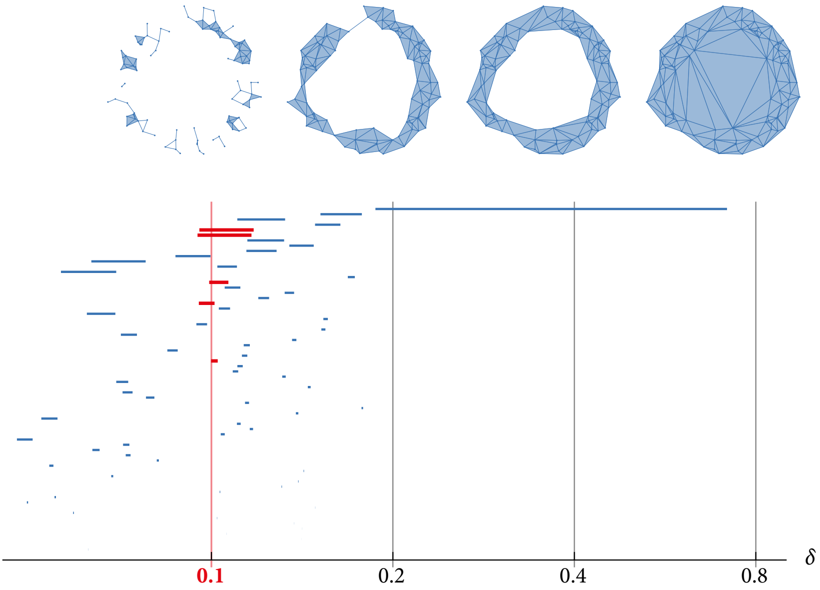

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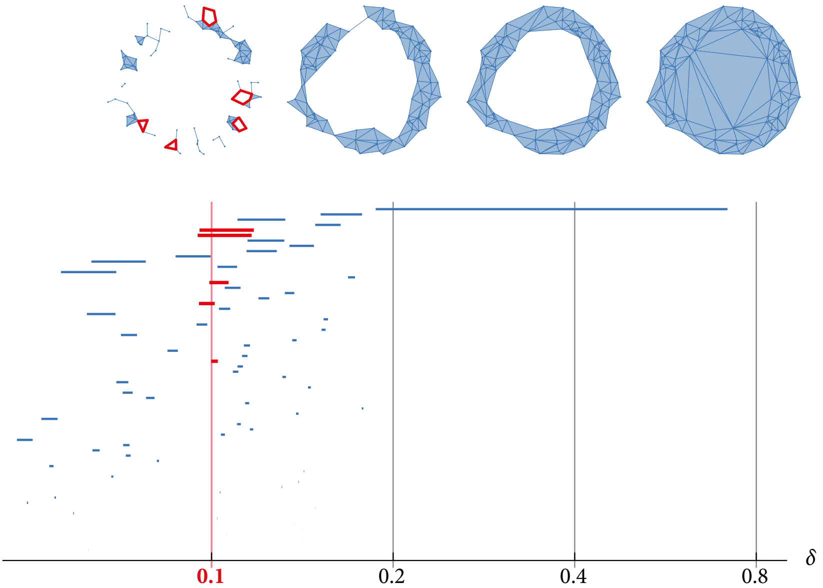

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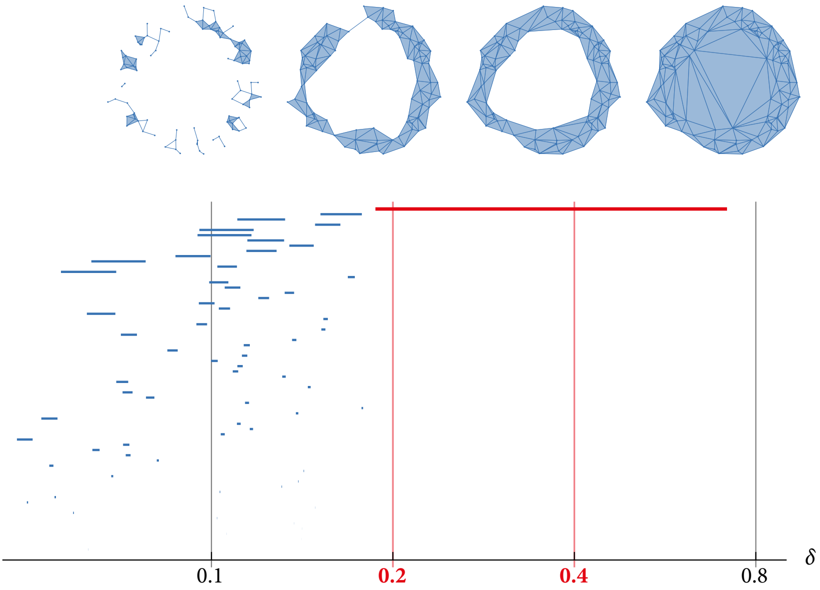

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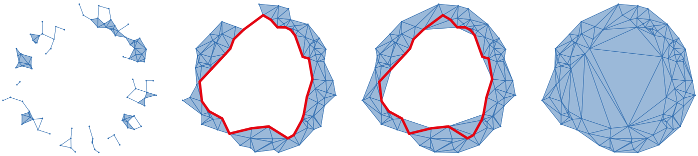

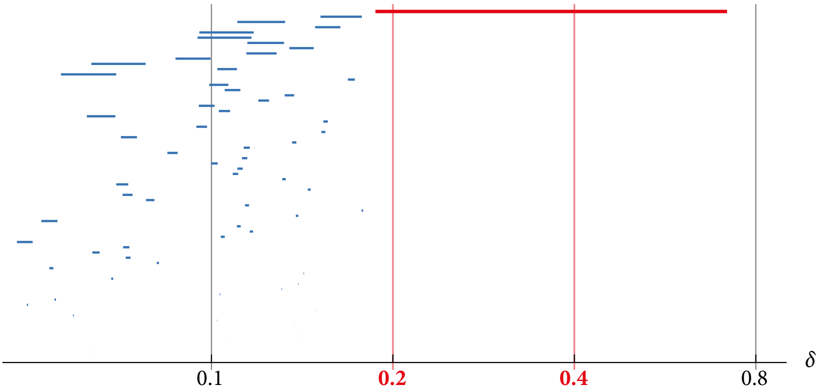

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  - $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$
  - A topological space K_t for each $t \in \mathbb{R}$
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- Consider homology with coefficients in a field (often $\mathbb{Z}_2$)
  $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M : \mathbf{R} \to \mathbf{Vect}$
  (persistence module)

Homology inference

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Requires strong assumptions:

Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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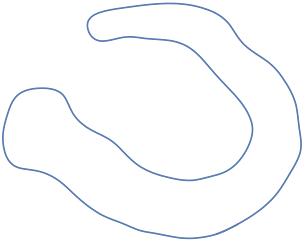

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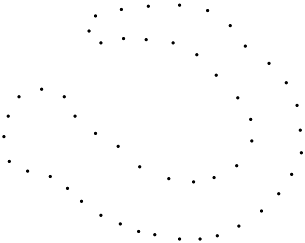

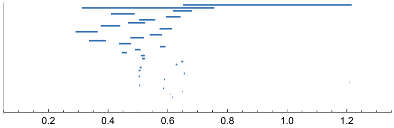

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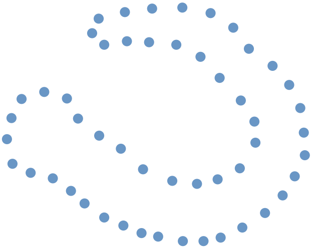

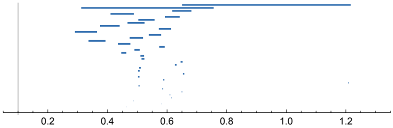

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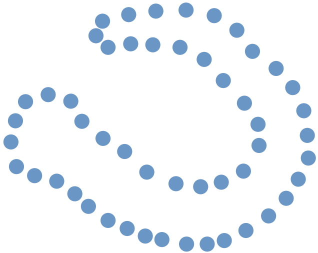

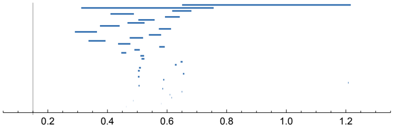

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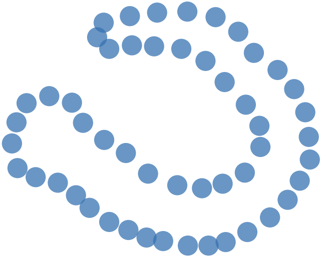

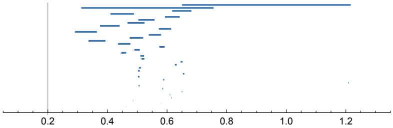

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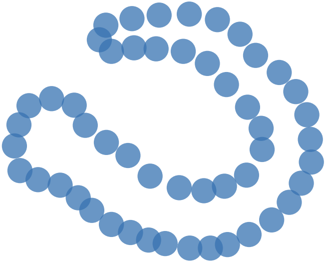

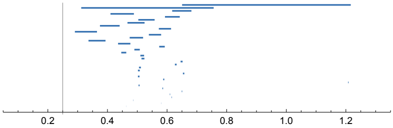

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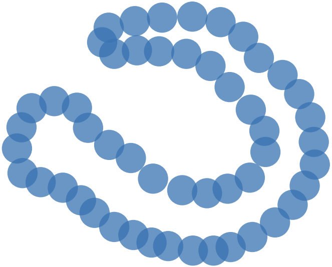

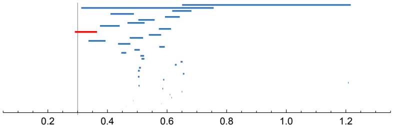

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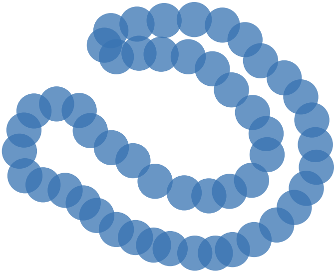

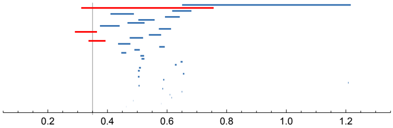

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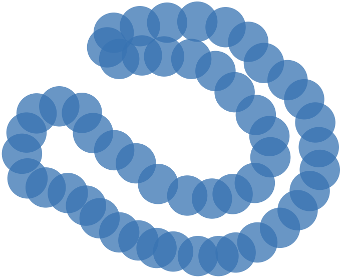

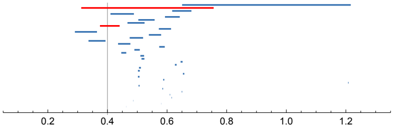

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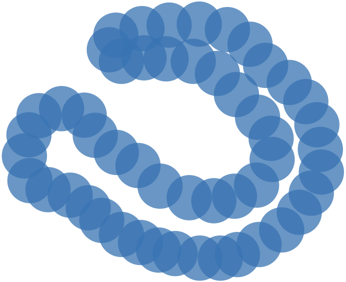

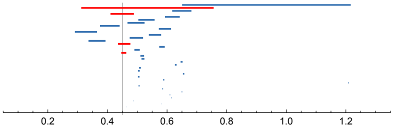

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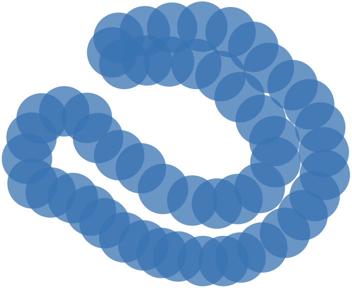

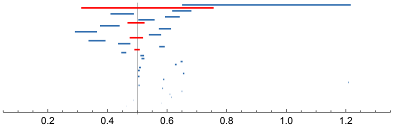

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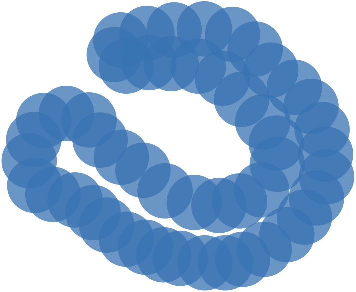

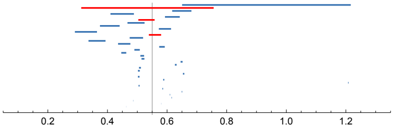

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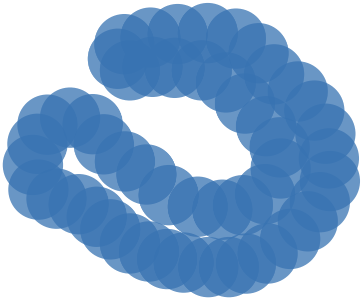

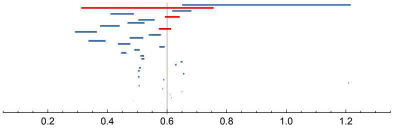

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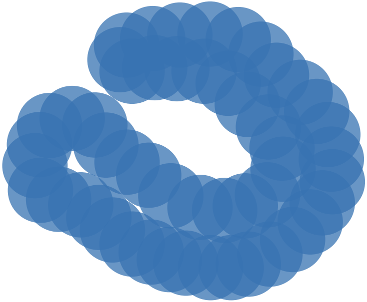

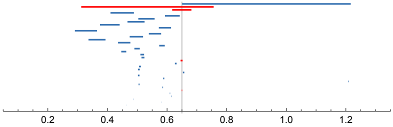

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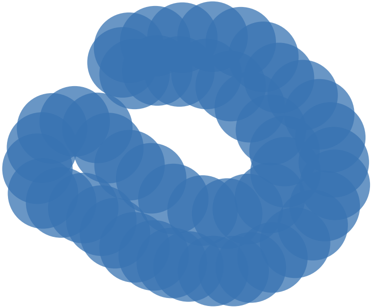

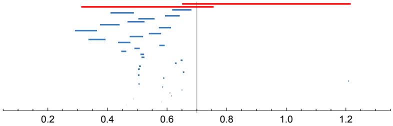

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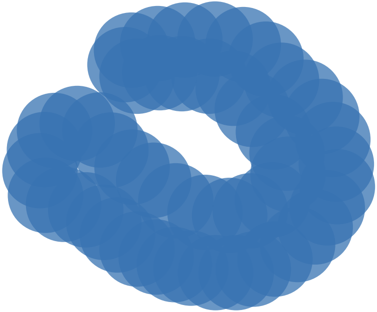

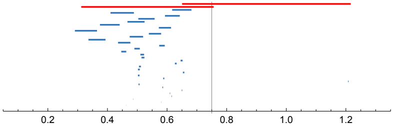

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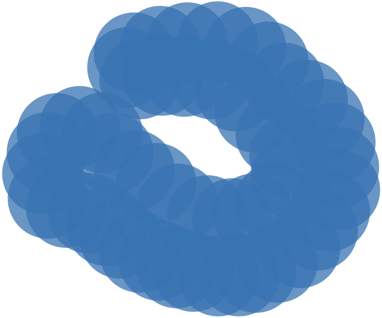

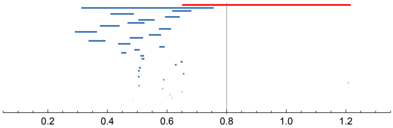

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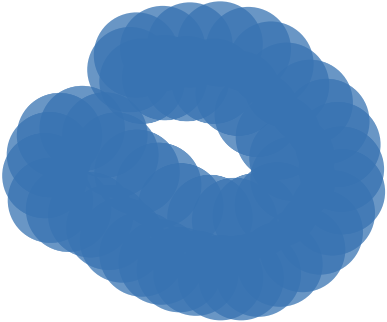

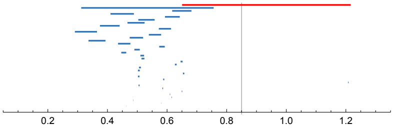

Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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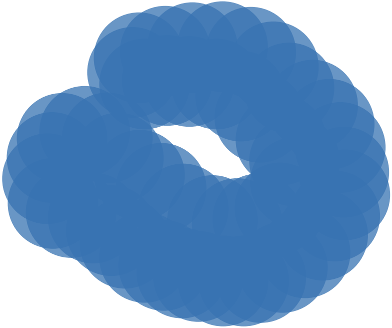
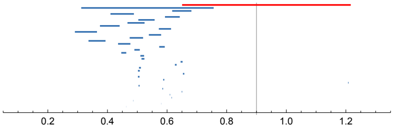




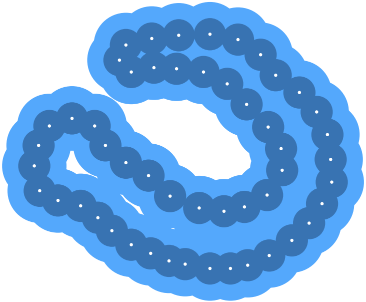
Homology inference using persistence

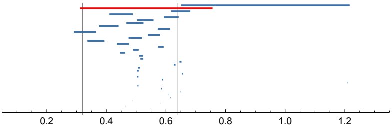
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. *Let* $P \subset \Omega$, $\delta > 0$ *be such that*

- $B_\delta(P)$ *covers* Ω, *and*
- *the inclusions* $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ *preserve homology.*

Then $H_*(\Omega) \cong \operatorname{im} H_*(B_\delta(P) \hookrightarrow B_{2\delta}(P))$.





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This motivates the *homological realization problem*:

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Given a simplicial pair $L \subseteq K$*, find* X *with* $L \subseteq X \subseteq K$ *such that*

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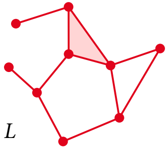

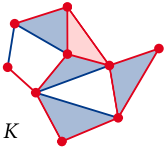

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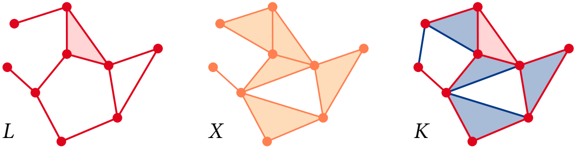

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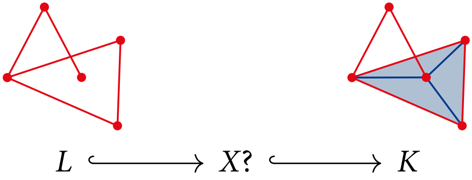

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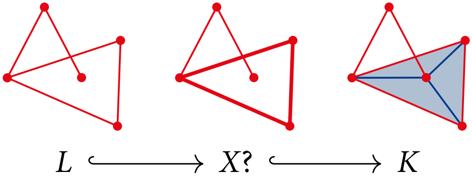

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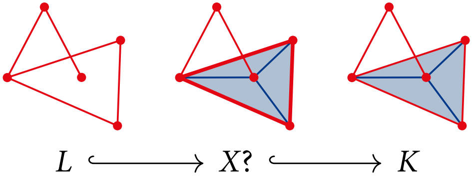

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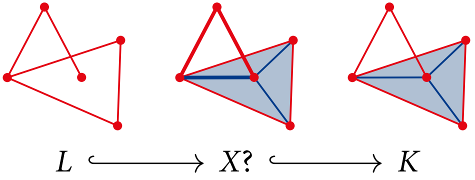

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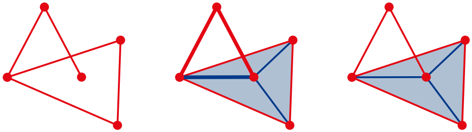

$$L \hookrightarrow X? \hookrightarrow K$$

Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Stability

Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

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↓	↓ structure theorem	
Combinatorics	intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$

Stability of persistence barcodes for functions

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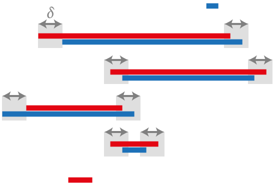

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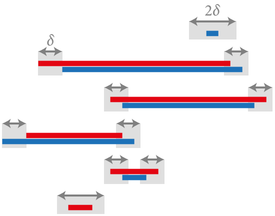

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Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ and $G_t = g^{-1}(-\infty, t]$.

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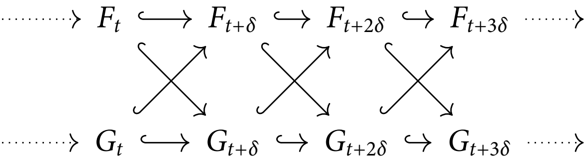

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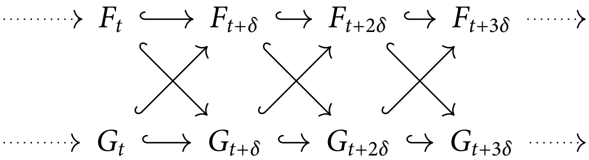
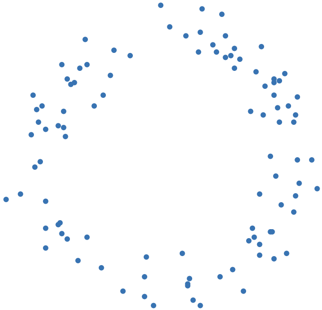


Applying homology (functor) preserves commutativity

- persistent homology of f, g yields
  δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Geometric interleavings



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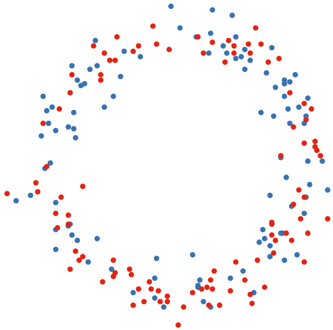

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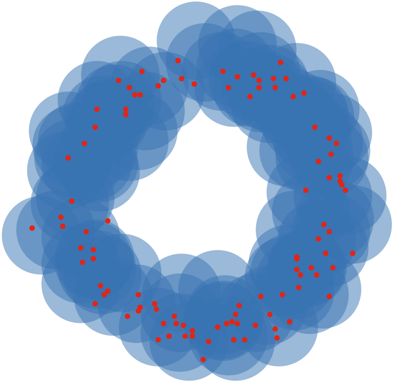

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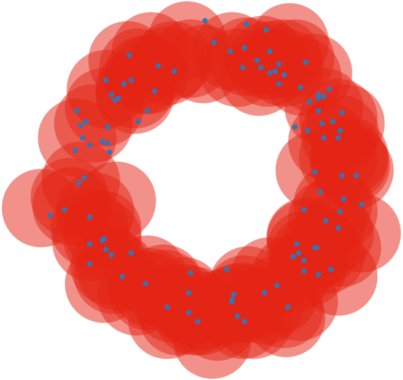

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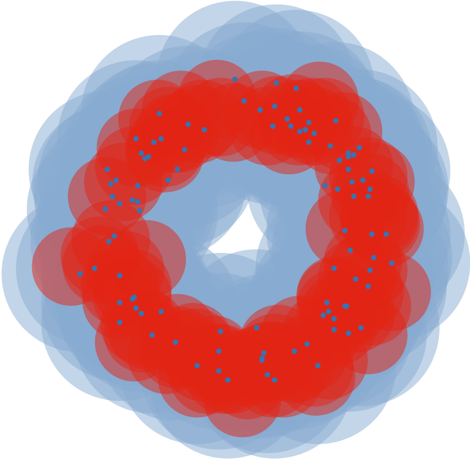

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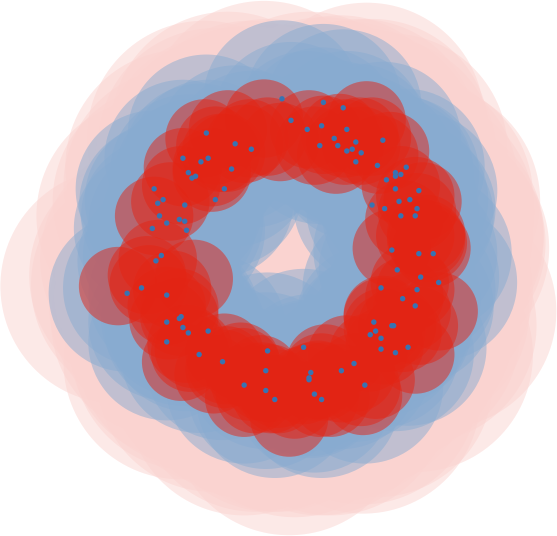

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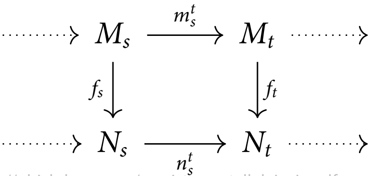

Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $\mathbb{K}(I)$ by

$$\mathbb{K}(I)_t = \begin{cases} \mathbb{K} & \text{if } t \in I, \\ 0 & \text{otherwise,} \end{cases}$$

with transition maps of maximal rank.

Schematic example:

$$\cdots\to 0 \longrightarrow 0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{K} \cdots\to \mathbb{K} \longrightarrow 0 \longrightarrow 0 \cdots\to$$

Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt; Crawley-Boewey 2015)

Let M be a pointwise finite-dimensional persistence module.

Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt; Crawley-Boewey 2015)

Let M be a pointwise finite-dimensional persistence module.
Then M is interval-decomposable:
there exists a unique collection of intervals $B(M)$ such that

$$M \cong \bigoplus_{I \in B(M)} \mathbb{K}(I).$$

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- The decomposition itself is not unique.
- This is why we use homology with coefficients in a field.

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then there exists a δ-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
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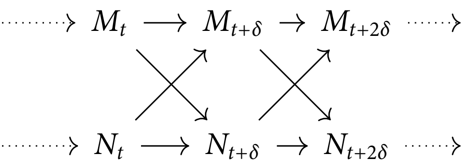
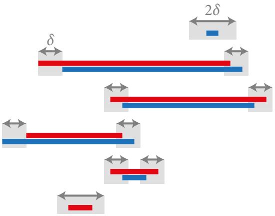
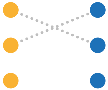




Barcodes as diagrams

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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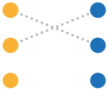
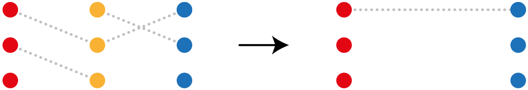


Composition of matchings $\sigma : S \nrightarrow T$ and $\tau : T \nrightarrow U$:



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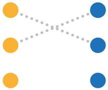

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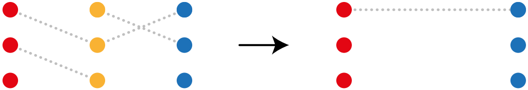
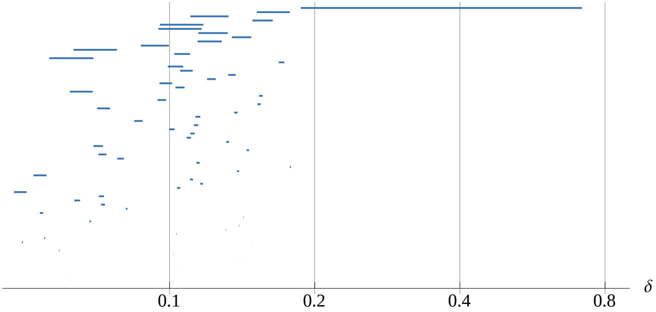


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode B as a functor $\mathbf{R} \to \mathbf{Mch}$:



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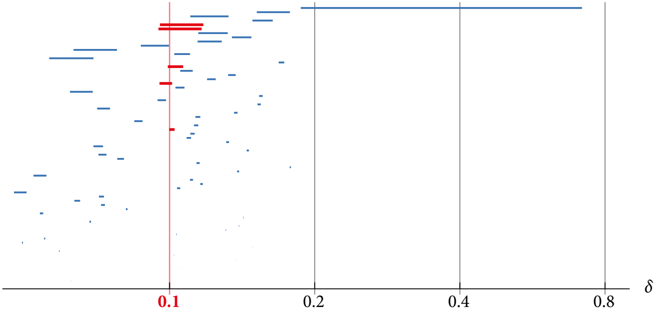

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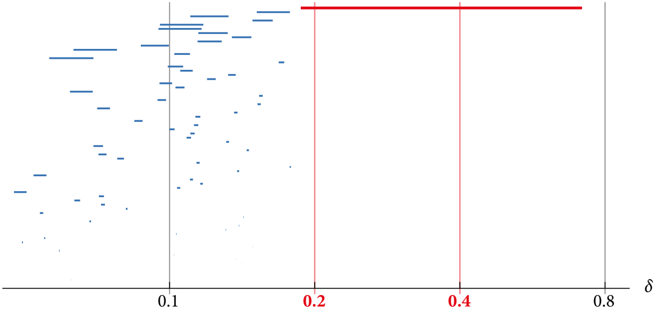

Stability via functoriality?

$$\begin{array}{ccccccc} F_t & & \hookrightarrow & & F_{t+2\delta} & & \\ & \searrow & & \nearrow & & \searrow & \\ & & G_{t+\delta} & & \hookrightarrow & & G_{t+3\delta} \end{array}$$

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Theorem (B, Lesnick 2015)

There exists no functor $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Mch}^{\mathbf{R}}$ *sending each persistence module to its barcode.*

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Proposition

There exists no functor $\mathbf{Vect} \to \mathbf{Mch}$ *sending each vector space of dimension* d *to a set of cardinality* d.

Induced barcode matchings

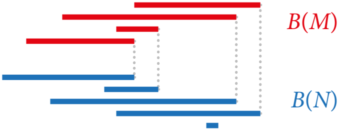
Structure of persistence submodules / quotients

Proposition

Let $f : M \to N$ be a monomorphism of persistence modules: each $f_t : M_t \to N_t$ is injective.

Then f induces an injective map $B(M) \to B(N)$ mapping each $I \in B(M)$ to some $J \in B(N)$ with larger or same left and same right endpoint.

$$\begin{array}{ccccc} \cdots\to & M_s & \longrightarrow & M_t & \cdots\to \\ & \downarrow & & \downarrow & \\ \cdots\to & N_s & \longrightarrow & N_t & \cdots\to \end{array}$$



Dually for epimorphisms (left and right exchanged).

Induced matchings

For a general morphism $f : M \to N$ of persistence modules: consider *epi-mono factorization*

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N.$$

Induced matchings

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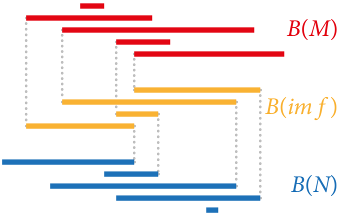
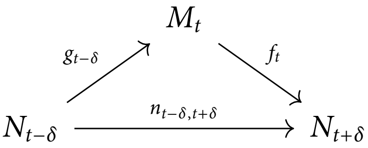
$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N.$$

- $\operatorname{im} f \hookrightarrow N$ induces injection $B(\operatorname{im} f) \hookrightarrow B(N)$
- $M \twoheadrightarrow \operatorname{im} f$ induces injection $B(\operatorname{im} f) \hookrightarrow B(M)$
- compose to a matching $B(M) \nrightarrow B(N)$:



Stability from interleavings

Consider interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(\forall t \in \mathbb{R})$:



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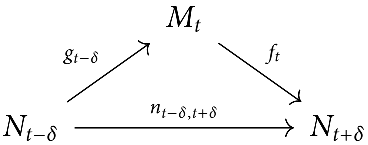

- $\operatorname{im} n_{t-\delta,t+\delta} \hookrightarrow \operatorname{im} f_t \hookrightarrow N_{t+\delta}$
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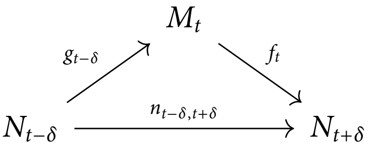

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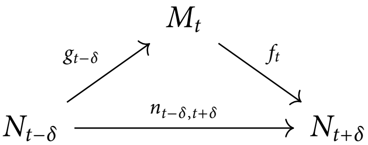

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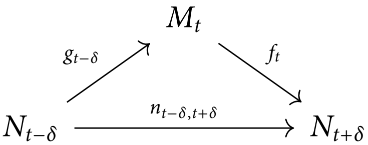

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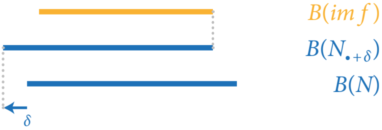

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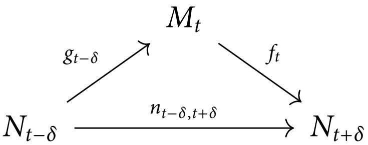

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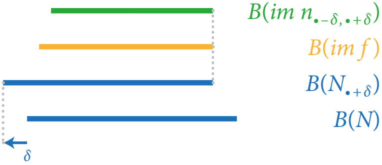

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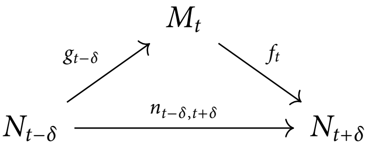

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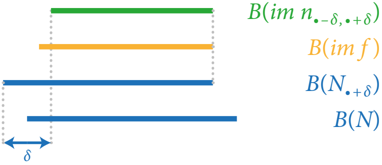

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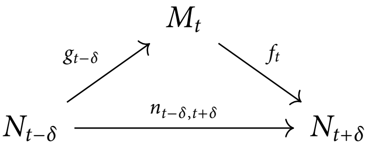

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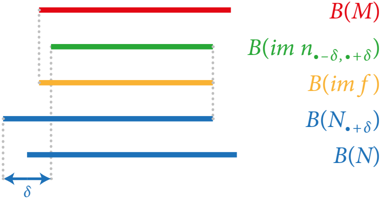

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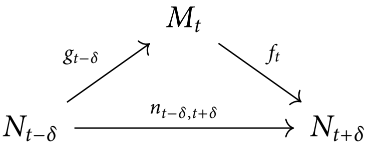

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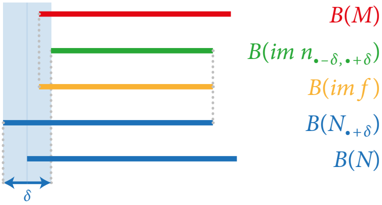

Sending persistence into Hilbert space

Extending the TDA pipeline

Mapping barcodes into a Hilbert space?

- desirable for (kernel-based) machine learning methods and statistics
- stability (Lipschitz continuity): important for reliable predictions
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Can we hope for something better?

No bi-Lipschitz feature maps for persistence

Theorem (B, Carrière 2018)

There is no bi-Lipschitz map from the persistence diagrams (with the interleaving or any p–Wasserstein distance) into any finite-dimensional Hilbert space, even when restricting to bounded range or number of bars.

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Theorem (B, Carrière 2018)

If there was such a bi-Lipschitz map into some Hilbert space, the ratio of the Lipschitz constants would have to go to ∞ together with the bounds on number or range of bars.

History

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Leray 1946]?

When was persistent homology invented first?

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ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are isolated;[1] the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally fulfilled. The generality of the theory rested upon the fact that the cases treated approximate in a certain sense the most general problems which it is

When was persistent homology invented first?

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Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory — ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
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Marston Morse and his mathematical works — ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
Cited by 24 Related articles All 4 versions Cite Save More

Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
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[PDF] **Persistence in discrete Morse theory** — psu.edu [PDF]
U Bauer - 2011 - Citeseer
The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [D] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

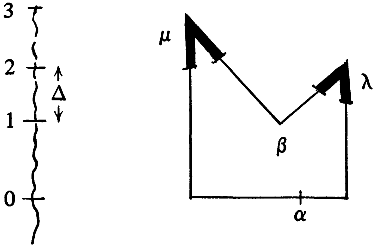


FIGURE 8

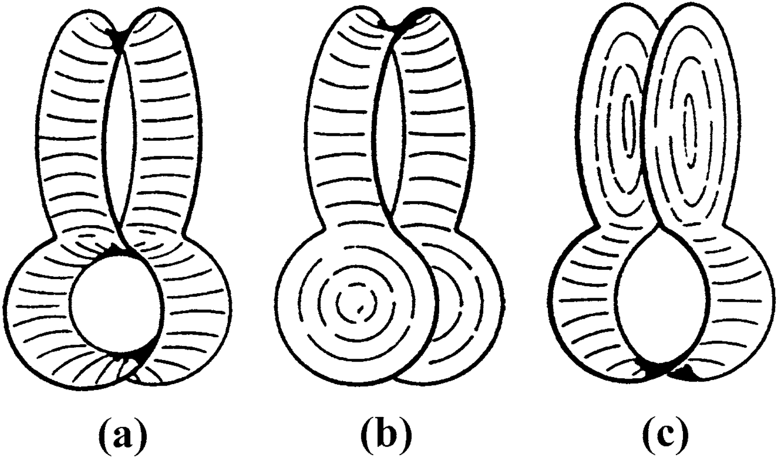
The values a where $H(a, a^-) \neq 0$ are indicated on the left, and correspond

Morse's *functional topology*

Key aspects:

- early precursor of persistence and spectral sequences
- uses Vietoris homology with field coefficients
- applies to a broad class of functions on metric spaces (not necessarily continuous)
- inclusions of sublevel sets have finite rank homology (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

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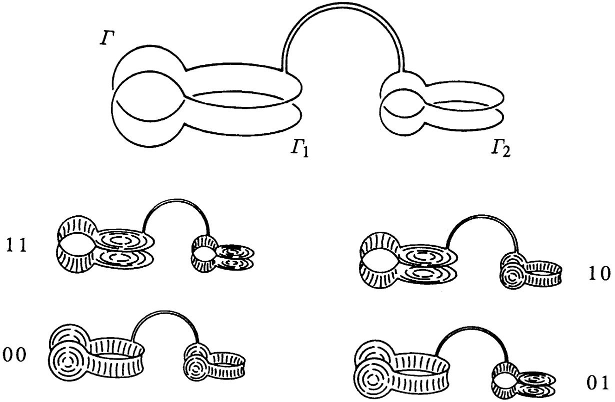

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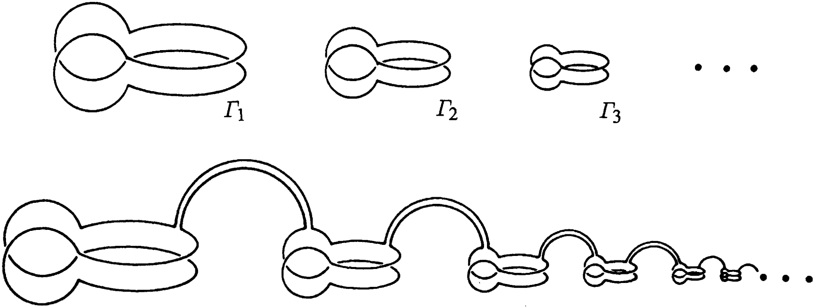

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Existence of unstable minimal surfaces

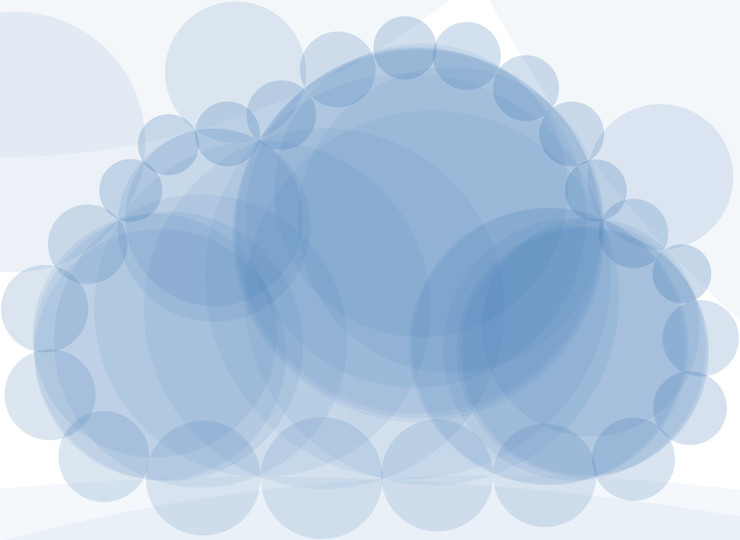
Using persistent homology:

- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

Theorem (Morse, Tompkins 1939)

There is a C_1 curve bounding an unstable minimal surface (an index 1 critical point of the area functional).

Thanks for your attention!



Thanks for your attention!