Multidimensional Persistence and Clustering

Magnus Bakke Botnan (TUM)

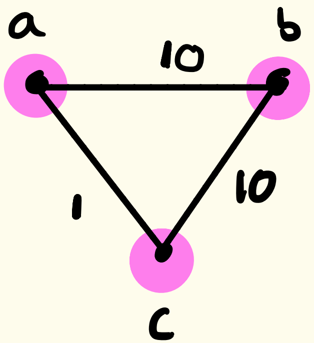
Ulrich Bauer (TUM), Steffen Oppermann (NTNU) and Johan Steen (NTNU)

Max Planck Institute, Leipzig, 19/02 2018

Theorem By Kleinberg

Let S be a finite set with $|S| \geq 2$.

A clustering function takes as input a distance $d: S \times S \to \mathbb{R}_{\geq 0}$ and returns a partition Γ of S.



$$\rightsquigarrow \{\{a, c\}, \{b\}\}$$

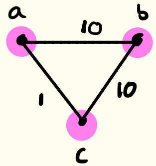
Theorem By Kleinberg

Let S be a finite set with $|S| \geq 2$.

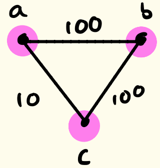
A clustering function f might satisfy:

Scale Invariance

$\forall \alpha > 0: \quad f(\alpha d) = f(d)$



$\rightsquigarrow \{\{a, c\}, \{b\}\}$



$\rightsquigarrow \{\{a, c\}, \{b\}\}$

Theorem By Kleinberg

Let S be a finite set with $|S| \geq 2$.

A clustering function f might satisfy:

Richness

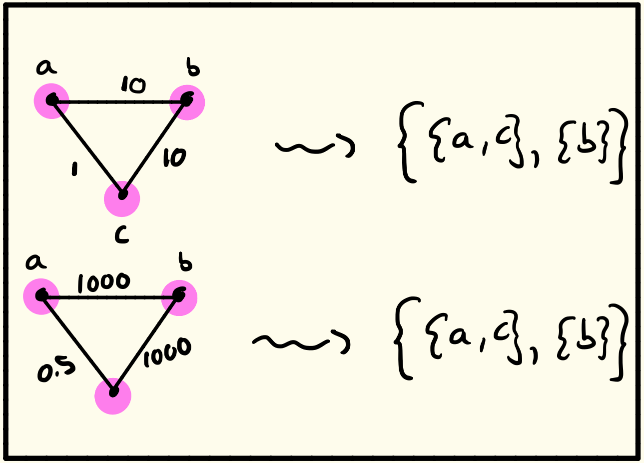
Let Γ be a partition of S.
There exists $d: S \times S \longrightarrow \mathbb{R}_{\geq 0}$
such that $f(d) = \Gamma$.

Theorem By Kleinberg

Let S be a finite set with $|S| \geq 2$.

A clustering function f might satisfy:

Consistency



Theorem By Kleinberg

Let S be a finite set with $|S| \geq 2$.
There exists no clustering function f on S that satisfies

i) Scale Invariance
ii) Richness
iii) Consistency

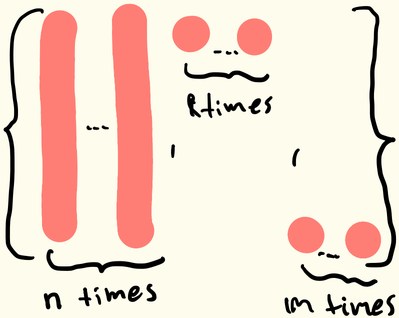
Consider two finite dimensional vector spaces V and W and a linear morphism $f : V \longrightarrow W$

Then f may be reduced to a matrix of the form

$$
\begin{array}{cc}
 & \begin{array}{cc} n & \ell \end{array} \\
\begin{array}{c} n \\ m \end{array} & \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right]
\end{array}
$$

In particular this means that we have the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\cong"'] \arrow[d, "f"'] & K^n \oplus K^\ell \oplus 0 \arrow[d, "1 \oplus 0 \oplus 0"] \\ W \arrow[r, "\cong"'] & K^n \oplus 0 \oplus K^m \end{tikzcd} \quad \longleftrightarrow$$



So we see that $f : V \to W$ decomposes into:

1. n copies of $K \xrightarrow{1} K$
2. ℓ copies of $K \to 0$
3. m copies of $0 \to K$

These intervals completely describe $f : V \to W$

More generally: Using similar arguemts one can show that any representation

$$V = V_0 \to V_1 \to V_2 \to \cdots \to V_n$$

decomposes into a direct sum of representations of the form

$$0 \to \cdots \to 0 \to \underbrace{k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k}_{[a \quad b]} \to 0 \to \cdots \to 0$$

Thm: V "$\cong$" collection of intervals.

Special Case: All Surjections.

1. $V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} V_n$

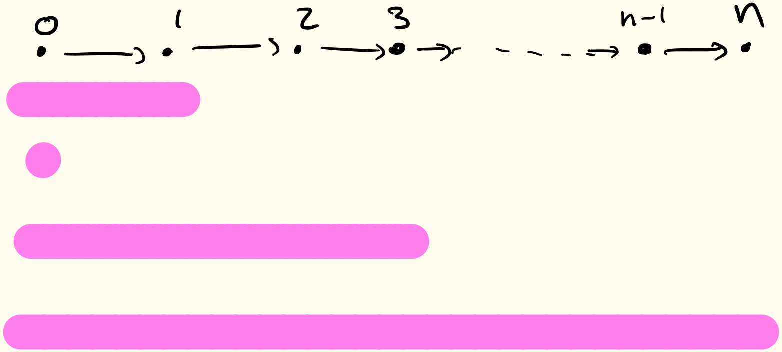
2. Assume we have decomposed
$V_{i+1} \xrightarrow{f_{i+1}} V_{i+2} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_{n-1}} V_n$

3. Extend this to a decomposition
   i) Let $\{a_1, \ldots, a_\ell\}$ be the basis for V_{i+1}
   ii) Choose a basis $\{z_1, \ldots, z_m\}$ for $\ker f_i \subseteq V_i$
   iii) Then $\{z_1, \ldots, z_m\} \cup \{z_{m+1} \in f_i^{-1}(a_1), \ldots, z_{m+\ell} \in f_i^{-1}(a_\ell)\}$
   is a basis for V_i

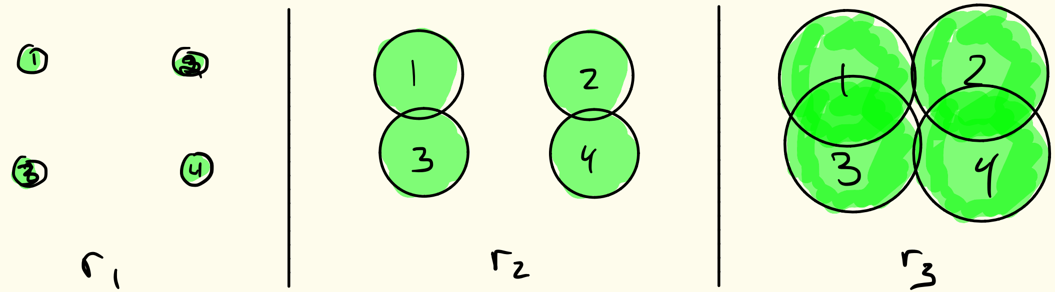
4. Iterate until $i=0$.

$$
\begin{tikzcd}
{} \arrow[r, two heads] & B_{n-2} & & \\
{} \arrow[r, two heads] & f_{n-2}^{-1}(B_{n-1}) \arrow[r, two heads] & B_{n-1} & \\
{} \arrow[r, two heads] & f_{n-2}^{-1}(f_{n-1}^{-1}(B_n)) \arrow[r, two heads] & f_{n-1}^{-1}(B_n) \arrow[r, two heads] & B_n
\end{tikzcd}
$$

Observation: every bar is supported on $i=0$.



Single Linkage Clustering



Apply 0-th homology

$$k^4 \xrightarrow{\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{array}} \!\!\!\!\twoheadrightarrow k^2 \xrightarrow{\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \end{array}} k$$

Theorem [Carlsson, Memoli]: SLC is uniquely "optimal"

optimal but....

Chaining Effect:

Single linkage clustering would fail to identify the two clusters

$\Downarrow$

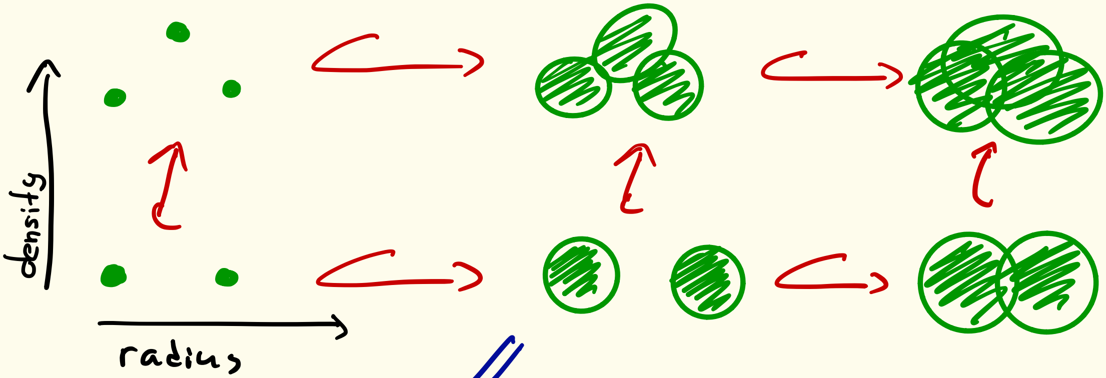
Not much used in practice

its linkage function.

A more principled way of taking density into account, that does not depend on ad hoc constructions which destroy the stability property, would be to explicitly build the density into the method. In Carlsson and Mémoli (2009) we study **multiparameter clustering methods***, which are similar to HC methods but we* track *connected components in a multiparameter landscape. We also study the classification and stability properties of multiparameter clustering methods.*

2D Clustering

Carlsson + Memoli '10
Multiparameter Hierarchical Clustering schemes



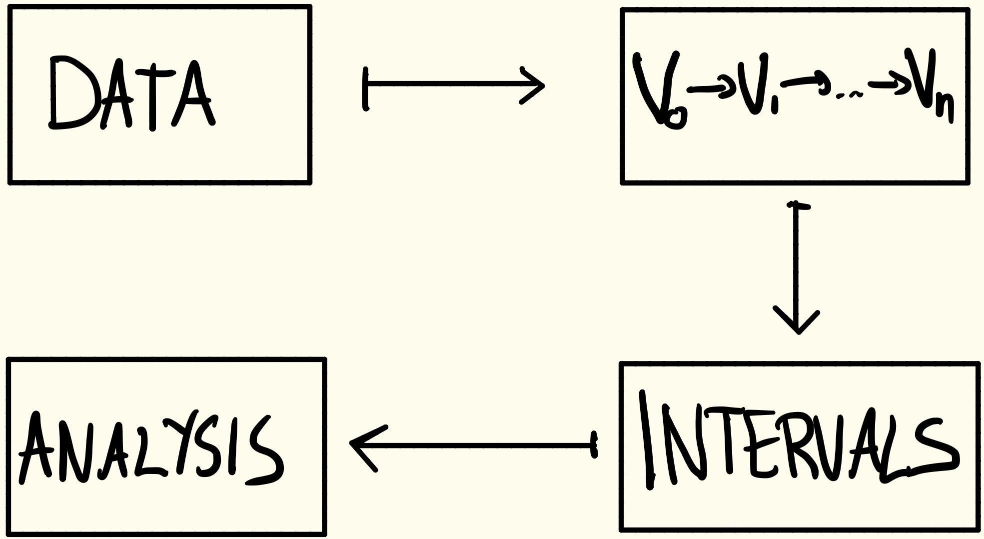
H_0

$$
\begin{tikzcd}
k^3 \arrow[r, two heads, "(1\,1\,1)"] & k \arrow[r, two heads, "1"] & k \\
k^2 \arrow[u, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] \arrow[r, two heads, "1"'] & k^2 \arrow[u] \arrow[r, two heads, "(1\,1)"'] & k \arrow[u, "1"']
\end{tikzcd}
$$

Note!

In this setup every horizontal morphism is an epimorphism.

TDA PIPELINE:



Poset Representations:

K alg. closed.

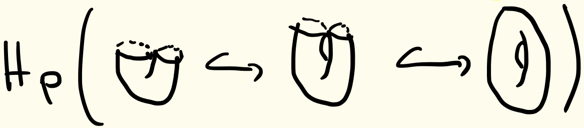
Let P be a finite poset.

A representation V of P is a collection of K-vector spaces $\{V_p\}_{p\in P}$ and linear morphisms $V_p \to V_q$ whenever $p \leq q$, such that

1. $V_p \to V_p$ is the identity.
2. $$\begin{tikzcd} V_p \arrow[r] \arrow[rr, bend left] & V_q \arrow[r] & V_{p'} \end{tikzcd}$$ commutes $\forall\ p \leq q \leq p'$

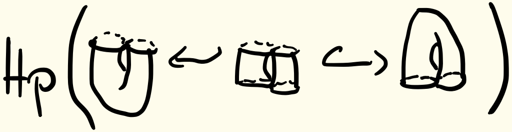
Linear

$$V_0 \longrightarrow V_1 \longrightarrow V_2$$



ZigZag

$$V_0 \longleftarrow V_1 \longrightarrow V_2$$



Circular

$$\begin{tikzcd}
 & V_0 & \\
V_1 \arrow[ur] \arrow[dr] & & V_3 \arrow[ul] \arrow[dl] \\
 & V_2 &
\end{tikzcd}$$

2D

$$\begin{tikzcd}
V_{20} \arrow[r] & V_{21} \arrow[r] & V_{22} \arrow[r] & V_{32} \\
V_{10} \arrow[r] \arrow[u] & V_{11} \arrow[r] \arrow[u] & V_{21} \arrow[r] \arrow[u] & V_{31} \arrow[u] \\
V_{00} \arrow[r] \arrow[u] & V_{10} \arrow[r] \arrow[u] & V_{20} \arrow[r] \arrow[u] & V_{30} \arrow[u]
\end{tikzcd}$$

Fact: Every V can be written as

$$V \cong \bigoplus_{i=1,\dots,k} V^i$$

where V^i is an indecomposable representation of P, in an essentially unique way

PROBLEM! What are the indecomposables?

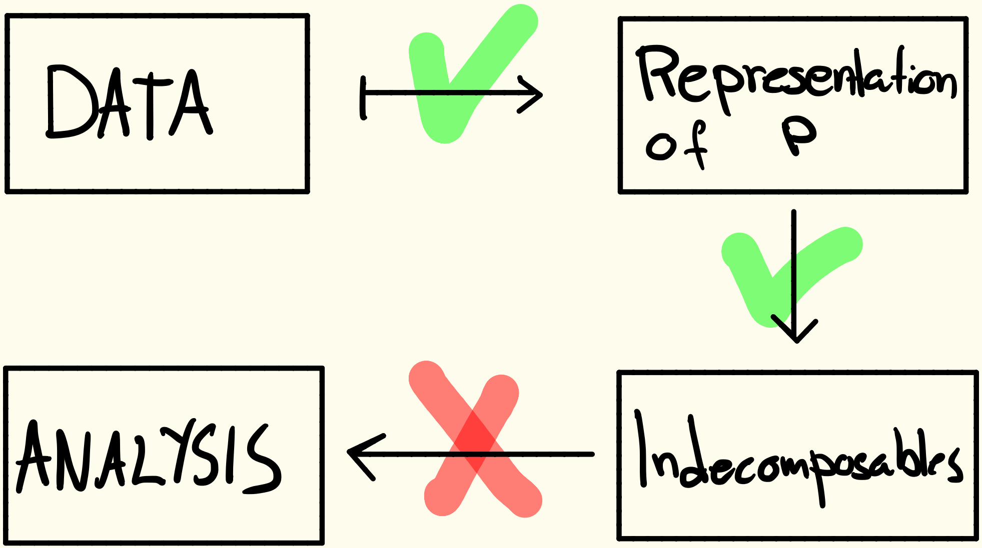
Decompositions

$$
\begin{tikzcd}
 & & K^2 \; e_3, e_4 \\
K \arrow[r, "1"] & K \arrow[ur, "\binom{1}{1}"] \arrow[dr, "1"'] & \\
e_1 & e_2 & K \; e_5
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
 & & K^2 \; e_3+e_4, e_4 \\
K \arrow[r, "1"] & K \arrow[ur, "\binom{1}{0}"] \arrow[dr, "1"'] & \\
e_1 & e_2 & K \; e_5
\end{tikzcd}
$$

$$
= \boxed{
\begin{tikzcd}
 & & K \\
K \arrow[r, "1"] & K \arrow[ur, "1"] \arrow[dr, "1"'] & \\
 & & K
\end{tikzcd}
}
\oplus
\boxed{
\begin{tikzcd}
 & & k \\
0 \arrow[r] & 0 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
}
$$

Indecomposable

TDA PIPELINE for general posets



Theorem (DROZD)

Let P be a finite poset. Then (up to iso.)

1. There exists a finite number of indecomposable representations of P. } Finite type

or

2. There exists a „canonical" way to parametrize the isomorphism classes in a sensibly finite way. } Tame

or

3. Impossible to understand the indecomposable representations } Wild

$$\begin{tikzcd} K \arrow[r, "1"] & K \arrow[r, "0"] & 0 \end{tikzcd}$$

$$\begin{tikzcd} V_0 \arrow[r] & V_1 \arrow[r] & V_2 \end{tikzcd}$$

Finite type

$$\begin{tikzcd} K & K \arrow[l, "1"'] \arrow[r, "0"] & 0 \end{tikzcd}$$

$$\begin{tikzcd} V_0 & V_1 \arrow[l] \arrow[r] & V_2 \end{tikzcd}$$

Finite type

Tame

$$\begin{tikzcd} & K^n & \\ K^n \arrow[ur, "J_n(t)"] \arrow[dr, "1"'] & & K^n \arrow[ul, "1"'] \arrow[dl, "1"] \\ & K^n & \end{tikzcd}$$

$$\begin{tikzcd} & V_0 & \\ V_1 \arrow[ur] \arrow[dr] & & V_3 \arrow[ul] \arrow[dl] \\ & V_2 & \end{tikzcd}$$

Wild

$$\begin{tikzcd} V_{20} \arrow[r] & V_{21} \arrow[r] & V_{22} \arrow[r] & V_{32} \\ V_{10} \arrow[r] \arrow[u] & V_{11} \arrow[r] \arrow[u] & V_{21} \arrow[r] \arrow[u] & V_{31} \arrow[u] \\ V_{00} \arrow[r] \arrow[u] & V_{10} \arrow[r] \arrow[u] & V_{20} \arrow[r] \arrow[u] & V_{30} \arrow[u] \end{tikzcd}$$

??

Multi-D

Let $P = [m] \times [n]$ be the grid poset on $m \cdot n$ vertices:

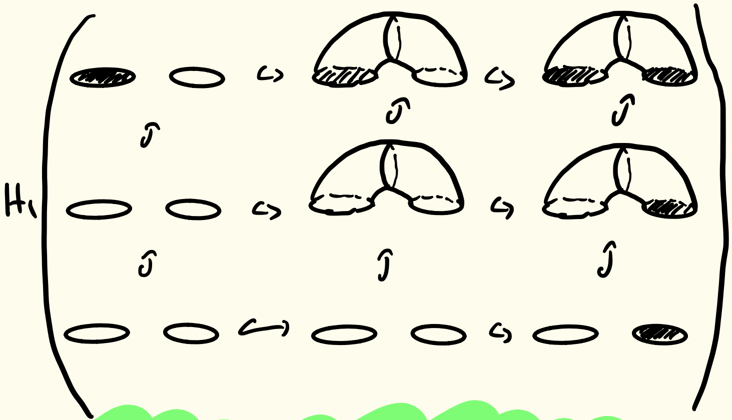
$$[3]\times[3] \quad = \quad \begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \bullet \arrow[u] \\
\bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \bullet \arrow[u]
\end{tikzcd}$$

Lemma (Carlsson+Zomorodian): For every $m, n \geq 1$ and any $V \in \mathrm{Rep}([m]\times[n])$ there exists a bifilatration of a CW-complex which yields V in p-th homology, $p \geq 0$.

Realization Example

$$
\begin{tikzcd}
k \arrow[r] & 0 \arrow[r] & 0 \\
k^2 \arrow[u, "(0\ 1)"] \arrow[r, "(1\ 1)"] & k \arrow[u] \arrow[r] & 0 \arrow[u] \\
K^2 \arrow[u, equal] \arrow[r, phantom, "="] & k^2 \arrow[u, "(1\ 1)"'] \arrow[r, "(1\ 0)"'] & k \arrow[u]
\end{tikzcd}
$$

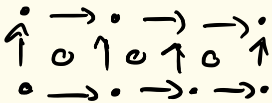
Indecomposable



Realization by a bifiltration

Not all wild!

- Hiroaka + Emerson proved using Auslander–Reiten theory that $[3]\times[2]$ and $[4]\times[2]$ are of finite type.



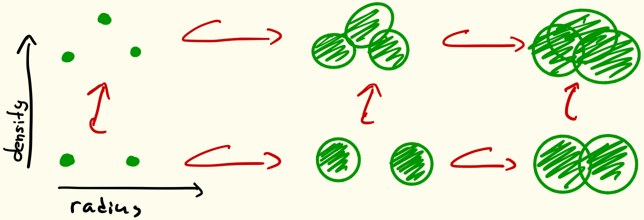
  This was applied to a problem in material science.

- The case $[5]\times[2]$ is known to be tame.
- For $n \geq 6$, $[n]\times[2]$ is wild.

- $[3]\times[3]$ is also tame.

Epimorphisms

Returning to our initial problem of clustering!



This type of clustering yields representations of the form:

$$\begin{tikzcd}
V_{1,2} \arrow[r, two heads] & V_{2,2} \arrow[r, two heads] & V_{3,2} \arrow[r, two heads] & \cdots \arrow[r, two heads] & V_{n,2} \\
V_{1,1} \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\circ"] & V_{2,1} \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\circ"] & V_{3,1} \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\circ"] & \cdots \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\circ"] & V_{n,1} \arrow[u]
\end{tikzcd}$$

All horizontal morphisms are epimorphisms!

$\boxed{\text{Maybe restricting to such representations will help?}}$

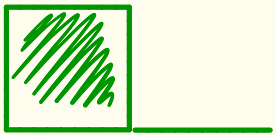
Epimorphisms

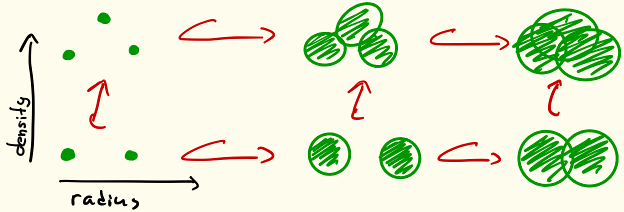
Define the following full subcategories of $\mathrm{Rep}([m]\times[n])$

- $\mathrm{Rep}^{\twoheadrightarrow}([m]\times[n])$, all horizontal morphisms epi
- $\mathrm{Rep}^{\twoheaduparrow}([m]\times[n])$, all vertical morphisms epi
- $\mathrm{Rep}^{\twoheaduparrow,\twoheadrightarrow}([m]\times[n])$ = Both vertical & horizontal epi.

<u>Lemma!</u> $\mathrm{Rep}^{\twoheadrightarrow}([m]\times 2)$ is of ==finite type.==

$$\begin{tikzcd} \bullet \arrow[r, two heads] & \bullet \arrow[r, two heads] & \bullet \\ \bullet \arrow[u] \arrow[r, two heads] & \bullet \arrow[u] \arrow[r, two heads] & \bullet \arrow[u] \end{tikzcd} \sim \begin{tikzcd} h \arrow[r] & h \arrow[r] & 0 \\ k \arrow[u, "i"] \arrow[r, "1"] & h \arrow[u, "1"'] \arrow[r, "1"] & h \arrow[u] \end{tikzcd} \sim$$





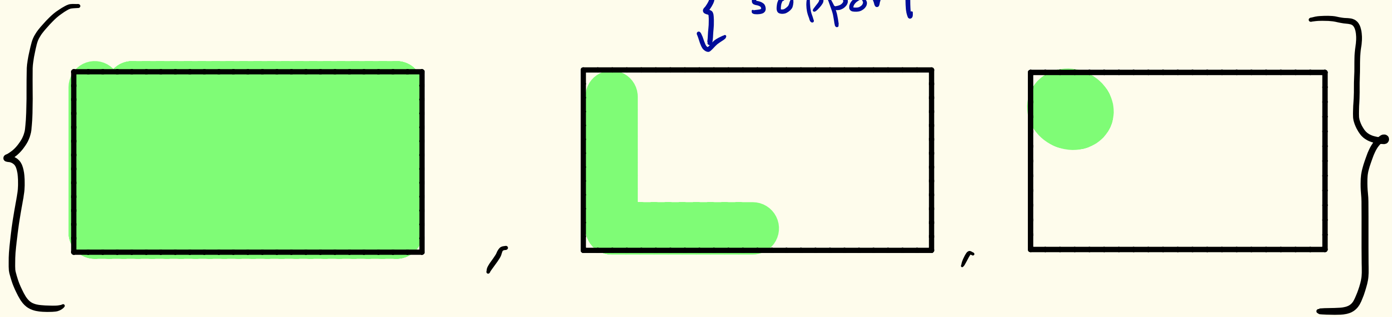
H_0 $\rightsquigarrow$

$$\begin{tikzcd} k^3 \arrow[r, two heads, "(1\,1\,1)"] & k \arrow[r, two heads, "1"] & k \\ k^2 \arrow[u, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] \arrow[r, two heads] & k^2 \arrow[u] \arrow[r, two heads, "(1\,1)"'] & k \arrow[u, "1"'] \end{tikzcd}$$

Decomposition

$$\begin{tikzcd} k \arrow[r, "1"] & k \arrow[r, "1"] & k \\ k \arrow[u, "1"] \arrow[r, "1"'] & k \arrow[u, "1"] \arrow[r, "1"'] & k \arrow[u, "1"] \end{tikzcd} \oplus \begin{tikzcd} k \arrow[r] & 0 \arrow[r] & 0 \\ k \arrow[u, "1"] \arrow[r, "1"] & k \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} k \arrow[r] & 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

support



Wild Things

"Same representation type"

Theorem:

$$\begin{array}{ccc} \mathrm{Rep}^{\Uparrow,\twoheadrightarrow}([m]\times[n]) & \sim & \mathrm{Rep}^{\Uparrow}([m]\times[n-1]) \\ \wr & & \wr \\ \mathrm{Rep}^{\twoheadrightarrow}([m-1]\times[n]) & \sim & \mathrm{Rep}([m-1]\times[n-1]) \end{array}$$

special case

Corollaries:

- $\mathrm{Rep}^{\twoheadrightarrow}([m]\times[2]) \sim \mathrm{Rep}([m]\times 1) \sim \mathrm{Rep}(A_m)$

  finite type

- $\mathrm{Rep}^{\twoheadrightarrow}([m]\times[3]) \sim \mathrm{Rep}([m]\times[2])$ which is finite type for $m \leq 4$.

- $\mathrm{Rep}^{\twoheadrightarrow}([m]\times[n])$ is wild for $n \geq 3$ and $m \geq 6$.

Note: Dual statements for monomorphisms.

$$F : \mathrm{Rep}^{\widehat{\Uparrow}}([m]\times[n])/\Pi \xrightarrow{\ \cong\ } \mathrm{Rep}([m-1]\times[n])$$

$$V \in \mathrm{Rep}^{\widehat{\Uparrow}}([m]\times[n]) \Longleftrightarrow$$

$$V = \begin{array}{ccccccc}
V_{1n} & \to & V_{2n} & \to & \cdots & \to & V_{mn} \\
\Uparrow & & \Uparrow & & & & \Uparrow \\
\vdots & & \vdots & & & & \vdots \\
\Uparrow & & \Uparrow & & & & \Uparrow \\
V_{12} & \to & V_{22} & \to & \cdots & \to & V_{m2} \\
\Uparrow & & \Uparrow & & & & \Uparrow \\
V_{11} & \to & V_{21} & \to & \cdots & \to & V_{m1}
\end{array}$$

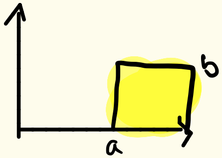
$$I^1 \xrightarrow{f_1} I^2 \xrightarrow{f_2} \cdots \xrightarrow{f_m} I^m$$

$$F(V) = F(I^1 \xrightarrow{f_1} \cdots \xrightarrow{f_m} I^m) =$$

$$\ker f_1 \to \ker(f_2\circ f_1) \to \cdots \to \ker(f_m\circ\cdots\circ f_1)$$

$$\Pi \doteq \mathrm{Rep}^{\widehat{\Uparrow}}([m]\times[n])$$

$\|$



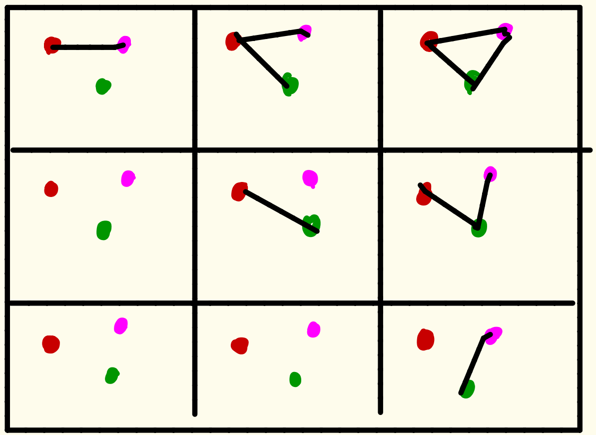
Intervals $[a,m]\times[1,b]$

Realizations

$V \in \mathrm{Rep}^{\uparrow\twoheadrightarrow}([m]\times[n])$. $\exists\ S : [m]\times[n] \longrightarrow CW$ such that

$$V \overset{?}{\cong} \tilde{H}_0 \circ S \quad ?$$

$$\begin{tikzcd}
k \arrow[r] & 0 \arrow[r] & 0 \\
k^2 \arrow[u, "(1\,0)"] \arrow[r, "(1\,1)"'] & k \arrow[u] \arrow[r] & 0 \arrow[u] \\
k^2 \arrow[u, equal, "1"] \arrow[r, equal, "1"'] & k^2 \arrow[u, "(1\,1)"'] \arrow[r, "(0\,1)"'] & k \arrow[u]
\end{tikzcd}
\quad \cong \quad \tilde{H}_0$$



Realizations:

Not always!

Lemma:

1) There are $V \in \mathrm{Rep}^{\widehat{\uparrow}\twoheadrightarrow}([m]\times[n])$ for which there is no $S : [m]\times[n]$ such that $V \ncong \widetilde{H}_0 \circ S$.

2) There exist indecomposables $V \in \mathrm{Rep}([m]\times[n])$ of arbitrary high dimension vector, such that

$$V \cong \widetilde{H}_0 \circ S$$

3) We can find an arbitrary number of non-isomorphic indecomposable $V_i \in \mathrm{Rep}^{\twoheadrightarrow}([m]\times[n])$ with the same dimension vector for which

$$V_i \cong \widetilde{H}_0 \circ S_i \qquad \forall i$$

Realization

open problem

- Let $V \in \mathrm{Rep}^{\hat{\uparrow} \twoheadrightarrow}([m]\times[n])$.
- Does there exist $S : [m]\times[n] \longrightarrow CW$ such that $H_0 \circ S \cong \bigoplus_i X_i$ and $V \cong X_j$ for some j?

In Practice

$V \in \mathrm{Rep}(Q)$ is thin if $\dim V_q \leq 1 \quad \forall q \in Q$

$$\begin{tikzcd} k \arrow[r] & 0 \\ k \arrow[u, "1"] \arrow[r, "1"'] & k \arrow[u, "1"'] \end{tikzcd}$$

Thin ✓

$$\begin{tikzcd} k^2 \arrow[r] & 0 \\ k^2 \arrow[u] \arrow[r] & k \arrow[u] \end{tikzcd}$$

Not Thin ✗

$$k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k$$

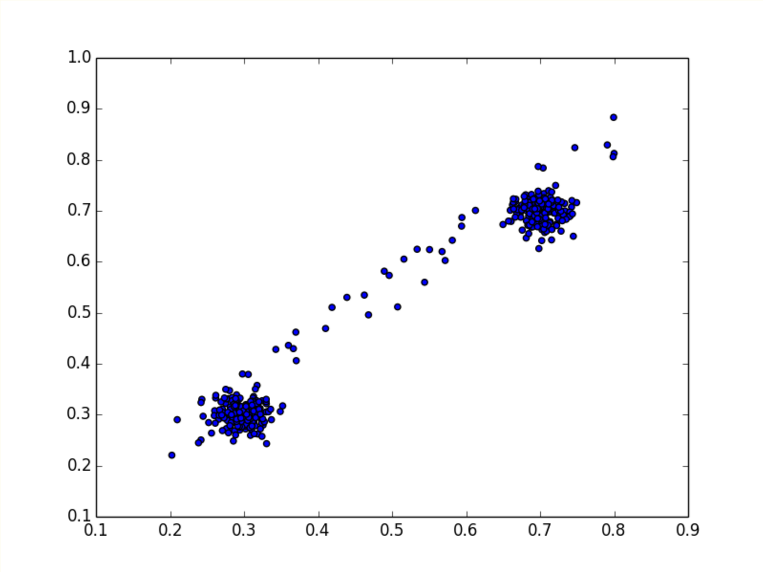
Thin ✓

Decompose $V \cong \bigoplus V_i$. If all V_i are thin (indecomposable) then V is completely described* by a collection of intervals in Q.

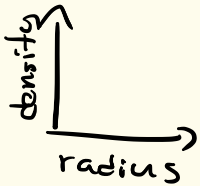
Problem: Given a random process which generates $V \in \mathrm{Rep}(Q)$. What is the probability that V has a non-thin summand?

* In Multi-D

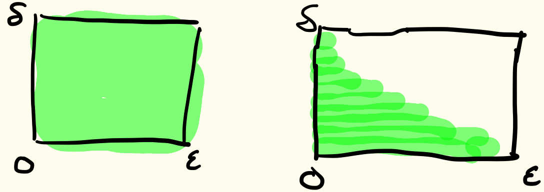
Clusters in $\mathbb{R}^2$



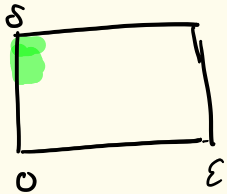
Filter by



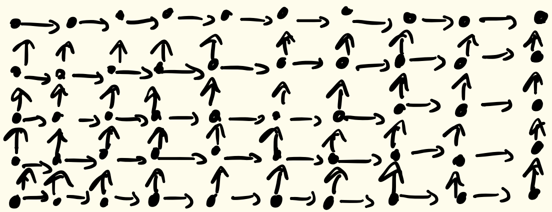
- All summands thin
- Two clear persistent intervals.



- Many small



Random Points



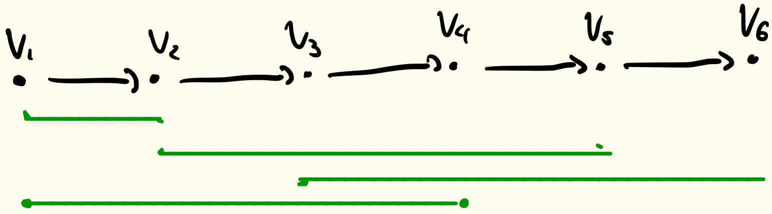
20 points in $\mathbb{R}^d$
10 distance scales
5 density scales

100 Runs

dimension d	2	50	200	1000
Non-thin M	14	42	56	50
Non-thin summands	15/1788	54/1742	60/1732	64/1372

Counting Bars

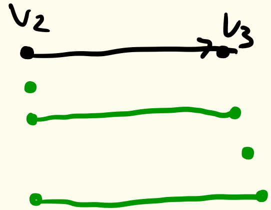
For 1D persistence:



Barcode

How many bars are supported on the whole of $[2,3]$?

Restrict the representation



2 bars.

Not hard to see: given an interval $[i,j]$, the number of bars supported on the whole interval $[i,j]$ equals $\mathrm{Rank}(V_i \longrightarrow V_j)$.

Counting Thin Summands

Let Q be a connected poset and $V \in Rep(Q)$ be indecomposable.

Note: V can be considered as a functor

$$V : Q \longrightarrow vec$$

Theorem:

$$\mathrm{Rank}(\lim V \to \mathrm{colim} V) = \begin{cases} 1 & \text{if } \dim V_i = 1 \ \forall i \text{ and every non-zero morphism is } 1. \\ 0 & \text{otherwise} \end{cases}$$

Example

i) lim = k

$$\begin{tikzcd} k \arrow[r, "1"'] & k \arrow[r, "1"] \arrow[dr, "1"'] & k \arrow[r, dashed, "1"] & \mathrm{colim} \cong k \\ & & k \arrow[ur, dashed, "1"'] & \end{tikzcd}$$

$\boxed{\text{Rank} = 1}$

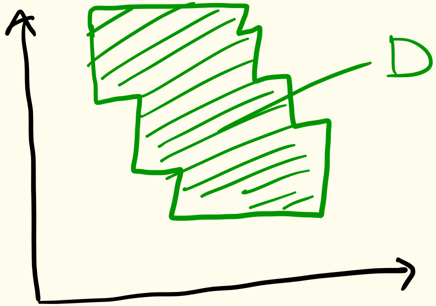
ii) lim = k

$$\begin{tikzcd} k \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & k^2 \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"'] & k \arrow[r, dashed, "0"] & \mathrm{colim} \cong 0 \\ & & k \arrow[ur, dashed, "0"'] & \end{tikzcd}$$

$\boxed{\text{Rank} = 0}$

Thin Summands in Multi-D

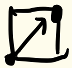
- Let $V \in \mathrm{Rep}([m]\times[n])$ and D a connected subposet.



- restricts to $M|_D \in \mathrm{Rep}(D)$.
- $V|_D \cong \bigoplus_j V_j$
- The number of j's such that $\dim((V_j)_i) = 1 \ \forall i \in Q_0$

  $= \mathrm{rk}(\lim V|_D \longrightarrow \mathrm{colim}\, V|_D)$.

- Can be efficiently computed using zigzag persistence.
- New interpretation of the rank invariant.



Thin Example

$$
\begin{tikzcd}
k \arrow[r] & 0 \arrow[r] & 0 \\
k^2 \arrow[u, "(1\,0)"] \arrow[r, "(1\,1)"] & k \arrow[u] \arrow[r] & 0 \arrow[u] \\
k^2 \arrow[u, equal] \arrow[r, equal] & k^2 \arrow[u, "(1\,1)"'] \arrow[r, "(0\,1)"'] & k \arrow[u]
\end{tikzcd}
$$

Indecomposable.

$\mathrm{Rank}(k^2 \longrightarrow 0) = 0.$

$$
\begin{tikzcd}
k^2 \arrow[r, "(1\,1)"] & k \\
k^2 \arrow[u, equal, "1"] \arrow[r, equal, "1"'] & k^2 \arrow[u, "(1\,1)"']
\end{tikzcd}
\cong
\boxed{
\begin{tikzcd}
K \arrow[r, "1"] & K \\
K \arrow[u, "1"] \arrow[r, "1"'] & K \arrow[u, "1"']
\end{tikzcd}
}
\oplus
\begin{tikzcd}
K \arrow[r] & 0 \\
K \arrow[u, "1"] \arrow[r, "1"'] & K \arrow[u]
\end{tikzcd}
$$

$\lim \longrightarrow \mathrm{colim}$

$=$

$k^2 \xrightarrow{(1\,1)} k$

$\mathrm{Rank} = 1$

Interleaving Distance

The interleaving distance is the most ← Lesnick ($\mathbb{K} = \mathbb{F}_p$)
discriminative <u>stable</u> distance on
multi-dimensional persistence modules

open problem: Computational complexity of computing $d_I(M,N)$.

partial Result:
s/w Håvard Bjerkevik

- NP-Hard for a generalization
- At least as hard as:

$$n\begin{bmatrix} 0 & 0 & 0 & ? \\ ? & ? & ? & ? \\ ? & ? & 0 & ? \\ 0 & ? & ? & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & ? & 0 \\ ? & 0 & 0 & ? \\ ? & ? & ? & ? \\ ? & 0 & ? & ? \end{bmatrix} n = I_n$$

$$\begin{array}{cc} \quad n & \quad n \end{array}$$

Example

i) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii) $$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & & \\ & & \\ & & \end{bmatrix}$$