What is p-adic geometry?

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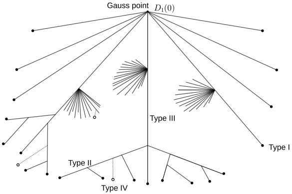

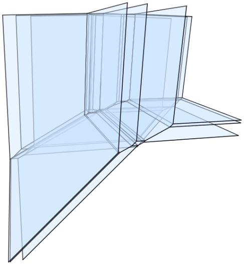

Non-Archimedean valued fields

Berkovich's theory

Huber's theory

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Non-Archimedean valued fields

A valued field is a field K with an absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying

- $|a| = 0$ if and only if $a = 0$,
- $|ab| = |a||b|$,
- $|a + b| \leq |a| + |b|$,

If $|a+b| \leq \max\{|a|, |b|\}$, then K is a non-Archimedean valued field. This condition is stronger than the triangle inequality.

The absolute value is called discrete if the value group $|K^*|$ is discrete in $\mathbb{R}_{>0}$.

Let's construct an example

Let p be a prime number.
We write $x \in \mathbb{Q}$ as $x = p^k \, {}^{a}\!/_{b}$, where p does not divide a and b, and $k \in \mathbb{Z}$.

$$\begin{array}{rccc} v_p : & \mathbb{Q} & \to & \mathbb{R} \cup \{\infty\} \\ & 0 & \mapsto & \infty \\ & x & \mapsto & k \end{array}$$

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$v_p : \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ satisfies the following property:

- $v_p(x) = \infty$ if and only if $x = 0$.
- $v_p(xy) = v_p(x) + v_p(y)$.
- $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$.

Such a map v_p is called a valuation.

p-adic numbers

We define the p-adic non-Archimedean absolute value on $\mathbb{Q}$,

$$|x|_p = p^{-v_p(x)}.$$

Example: $|3-2|_5 = 1$ $\qquad |27-2|_5 = \frac{1}{25}$

The field of p-adic numbers $\mathbb{Q}_p$ is the completion of the field of rational numbers $\mathbb{Q}$ under the p-adic absolute value.

Completion:

- absolute value $|\cdot|_p$ uniquely extends to $\mathbb{Q}_p$.
- $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.
- $\mathbb{Q}_p$ is complete.

Ostrowski's Theorem

Remark: The field of real numbers $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the (Archimedean) absolute value $|\cdot|_\infty$.

The absolute values $|\cdot|_\infty$, $|\cdot|_p$ and $|\cdot|_q$, with $p \neq q$, are not equivalent (just think of the sequence $\{p^n\}_n$).

Theorem (Ostrowski 1916)

Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_\infty$ or $|\cdot|_p$ for some prime number p.

Theorem (Hasse–Minkowski Theorem 1921)

A quadratic form Q with coefficients in $\mathbb{Q}$ has a non-trivial solution in $\mathbb{Q}$ if and only if Q has a non-trivial solution in $\mathbb{R}$ and in $\mathbb{Q}_p$ for every p.

More Examples

- Field K with trivial absolute value.
- The field $\mathbb{C}\{\{t\}\}$ of Puiseux series with complex coefficients:

$$x = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots,$$

  where $a_1 < a_2 < a_3 < \cdots$ are rational numbers that have a common denominator and $c_1 \neq 0$.

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- Number fields, i.e., finite field extensions of $\mathbb{Q}$.
- The completion of $\mathbb{Q}_p(p^{\frac{1}{p^\infty}}) = \cup_{n\geq 1}\mathbb{Q}_p(p^{\frac{1}{p^n}})$ (... perfectoid field, wait for the next lecture!).

Valuation ring and residue field

Let K be a non-Archimedean field. The valuation ring is

$$R = \left\{x \in K \,\middle|\, |x| \leq 1\right\}.$$

It is a local ring with maximal ideal

$$\mathcal{M} = \left\{x \in R \,\middle|\, |x| < 1\right\}.$$

The residue field k is the quotient $R/\mathcal{M}$.

Example:

- $K = \mathbb{Q}_p$, then $k = \mathbb{Z}/p\mathbb{Z}$.
- $K = \mathbb{C}\{\{t\}\}$, then $k = \mathbb{C}$.

Topological properties of NA fields

Remark that if $|a| \neq |b|$, then $|a+b| = \max\{|a|, |b|\}$.

- $D_r(a) = \{x \in \mathbb{Q}_p \,:\, |x-a| \leq r\}$
- $D_r(a)^\circ = \{x \in \mathbb{Q}_p \,:\, |x-a| < r\}$
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We have the following properties:

- $D_r(b)^\circ \subset C_r(a)$ for every $b \in C_r(a)$.
- Closed disks are open: $D_r(a) = D_r(a)^\circ \cup C_r(a)$.
- Every point in a disk is a center: $D_r(x) = D_r(y)$ for every $y \in D_r(x)$.
- The field K is Hausdorff and locally compact, but totally disconnected.

What goes wrong

Let K be a complete non-trivial NA valued field.

A K-manifold is a space that "looks like patches of K^n".

K is totally disconnected, then X is totally disconnected.

A function is analytic if it can be given locally by convergent power series.
K is totally disconnected, there are too many analytic functions.

We want a better space!

Non-Archimedean valued fields

Berkovich's theory

Huber's theory

Berkovich affine line

Let A be a ring. A multiplicative seminorm on A is a multiplicative function $|\cdot| : A \to \mathbb{R}_{\geq 0}$ such that

- $|0| = 0$ and $|1| = 1$,
- $|f + g| \leq |f| + |g|$

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Let K be a valued field. We define $\mathbb{A}_K^{1,\mathrm{Berk}}$ as follows:

- points $\rightsquigarrow$ seminorms on $K[T]$ which extend the absolute value on K
- topology $\rightsquigarrow$ weakest topology such that for every $f \in K[T]$ the map $|\cdot| \to |f|$ is continuous.

Points on $\mathbb{A}_K^{1,\mathrm{Berk}}$

Multiplicative seminorm $|\cdot|_x$ gives a point $x \in \mathbb{A}_K^{1,\mathrm{Berk}}$.

- $K = \mathbb{C}$
  The seminorms on $\mathbb{C}[T]$ are of the form $f \to |f(z)|$, for some $z \in \mathbb{C}$. So $\mathbb{A}_{\mathbb{C}}^{1,\mathrm{Berk}}$ is homeomorphic to $\mathbb{C}$. Seminorms correspond to maximal ideals of $\mathbb{C}[T]$.

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  We have more seminorms:

  $$|f|_{D_r(a)} = \sup_{x \in D_r(a)} |f(x)|, \text{ with } D_r(a) = \{x \in K \, : \, |x - a| \leq r\}.$$

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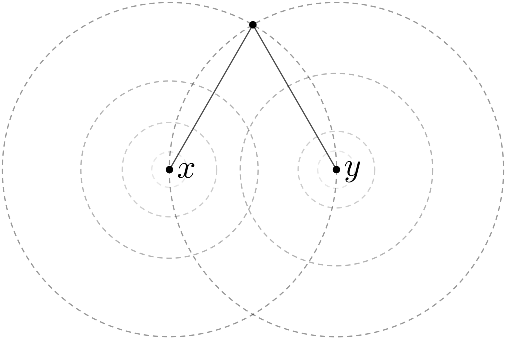


$$D_0(x) \rightsquigarrow D_r(x) \rightsquigarrow D_{|x-y|}(x) = D_{|x-y|}(y) \rightsquigarrow D_r(y) \rightsquigarrow D_0(y)$$

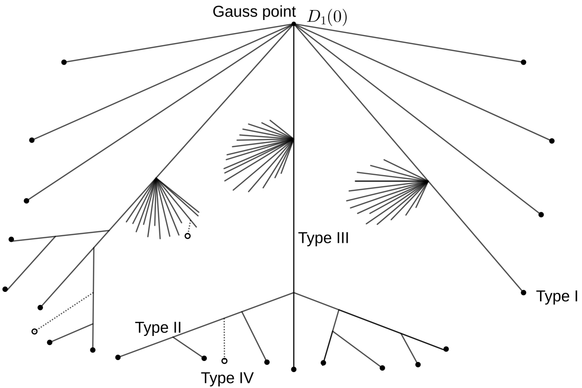
Berkovich's Classification Theorem

Theorem (Berkovich's Classification Theorem)

Let K be an algebraically closed, complete non-Archimedean field. Every point $x \in \mathbb{A}_K^{1,Berk}$ corresponds to a decreasing nested sequence $\{D_{r_i}(a_i)\}$ of closed discs. Classification is done according to $D = \cap_i D_{r_i}(a_i)$.

- *TYPE I: $D = D_0(a)$.*
- *TYPE II: $D = D_r(a)$, with $r \in |K^*|$.*
- *TYPE III: $D = D_r(a)$, with $r \notin |K^*|$.*
- *TYPE IV: $D = \emptyset$.*

$\mathbb{A}_K^{1,\mathrm{Berk}}$

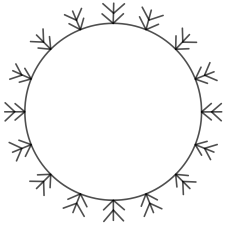


- Points of Type I are dense in $\mathbb{A}_K^{1,\mathrm{Berk}}$.
- $X_K^{1,\mathrm{Berk}}$ is uniquely path connected.
- Type II points are branching points.

Analytification of affine algebraic varieties

Let $X = \mathrm{Spec} K[T_1, \ldots, T_n]/I$. The analytification X_K^{Berk} is the set of semimorms on $K[T_1, \ldots, T_n]/I$ extending the absolute value on K.

- X connected iff X^{Berk} path connected
- X separated iff X^{Berk} Hausdorff
- X proper iff X^{Berk} compact
- same dimensions
- ...



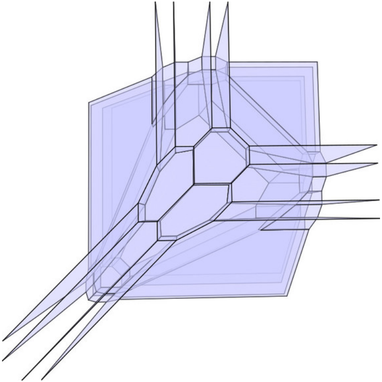
A tropical break

Let I be an ideal in $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ and X its variety on the algebraic torus $\mathbb{T}^n$. The tropical variety $\mathrm{Trop}(X)$ is defined as

$$\text{the closure in } \mathbb{R}^n \text{ of } \left\{ (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \,\middle|\, (x_1, x_2, \ldots, x_n) \in X \right\}.$$

The tropical variety $\mathrm{Trop}(X)$ is a pure rational polyhedral complex.

- Enumerative geometry
- Brill-Noether theory of curves
- Mirror symmetry
- Optimization
- ...



Analytification and Tropicalization

Let X be an affine algebraic variety over K, and $\varphi : X \to \mathbb{A}^m$ an affine embedding.

The tropicalization $\mathrm{Trop}(X, \varphi)$ of X depends on φ.

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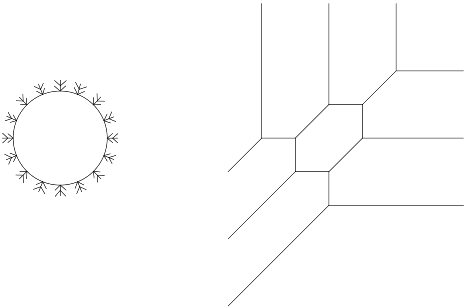
Theorem (Payne '09)

The analytification X^{an} is homeomorphic to the limit of all tropicalizations $Trop(X, \varphi)$.



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Higher rank valuations

Let A be a ring. A valuation is a map to a totally ordered abelian group Γ, $|\cdot| : A \to \Gamma \cup \{0\}$, satisfying the following properties:

- $|0| = 0$, $|1| = 1$,
- $|x + y| \leq \max\{|x|, |y|\}$.

Higher rank valuations

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- $|0| = 0$, $|1| = 1$,
- $|x + y| \leq \max\{|x|, |y|\}$.

Example: $A = K\langle T\rangle = \left\{ \sum_{a_n \geq 0} c_n T^{a_n} \right\}$ the ring of convergent power series on $|T| \leq 1$.
$A^+ = R\langle T\rangle$ the subspace of power series in the valuation ring R.
We define $X = \mathrm{Spa}(A, A^+)$ as

$$\left\{ |\cdot| : A \to \Gamma \cup \{0\} \text{ continuous valuation on } A \,\middle|\, |f| \leq 1, f \in A^+ \right\}$$

Points on $\mathrm{Spa}(A, A^+)$

- TYPE I: $f \mapsto |f(a)|$, $|a| \leq 1$.

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- TYPE IV: as before for not spherically complete field

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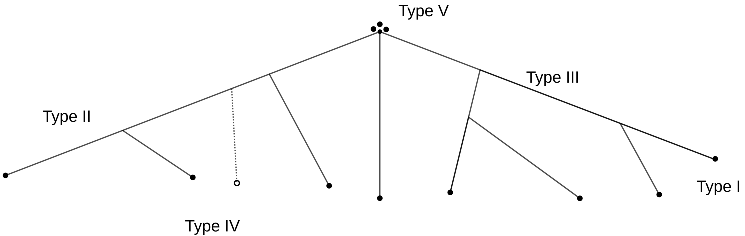
- TYPE IV: as before for not spherically complete field
- TYPE V: Let $|a| \leq 1$ and $0 < r \leq 1$. Let γ be a number infinitesimally smaller or bigger than r and $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$.

$$f = \sum c_n (T - a)^n \mapsto \max |c_n| \gamma^n.$$

If $r \notin |K^*|$, then these points are equivalent to point III.



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