Ramification and Perfectoid fields

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Local fields

- $\mathbb{Q} \dashrightarrow \mathbb{R}$ by completion (equivalence classes of "limits" of cauchy sequences)
- Fix p prime
    - p-adic absolute value: $0 \neq x \in \mathbb{Q}$
    - $\exists! n \in \mathbb{Z}$: $x = p^n \cdot \frac{a}{b}$ where p divides neither a or b.
    - set $|x|_p = p^{-n}$ $(|0|_p = 0)$
    - non-archimedian absolute value
- Ostrowski: every non-trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime p or the archimedian absolute value.

Local fields

- $\mathbb{Q}_p =$ completion of $\mathbb{Q}$ wrt. $|\cdot|_p$

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{\infty} a_i p^i \mid k \in \mathbb{Z}, a_i \in \{0, ..., p-1\}, a_k \neq 0 \right\}$$

- $\mathbb{Z}_p$ ring of integers $\subseteq \mathbb{Q}_p$

$$\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$$

- Discrete valuation ring (exactly one non-zero prime ideal $\mathfrak{p}_{\mathbb{Q}_p}$)
- $\kappa(\mathbb{Q}_p) \cong \mathbb{F}_p$ finite residue field of characteristic $\chi(\mathbb{Q}_p) = p$
- local field of mixed characteristic

Local fields

- $\mathbb{F}_p((t))$ power series in an indeterminate t:

$$\mathbb{F}_p((t)) = \left\{ \sum_{i=k}^{\infty} a_i t^i \mid k \in \mathbb{Z}, a_i \in \{0, ..., p-1\} = \mathbb{F}_p, a_k \neq 0 \right\}$$

- local field of equal characteristic p
- formally similar elements but different operations!

  $\Rightarrow$ similarities?

What is ramification?

- Extension of local fields $\mathbb{Q}_2(i)/\mathbb{Q}_2$
- $\mathfrak{p} = 2\mathbb{Z}_2$ and $\mathfrak{P} = (1-i)\mathbb{Z}_2[i]$ correspondig prime ideals
- Consider extension of primes:

$$\mathfrak{p}\mathbb{Z}_2[i] = 2\mathbb{Z}_2[i] = (1+i)(1-i)\mathbb{Z}_2[i] = \mathfrak{P}^2$$

- corresponding exponent $e(\mathbb{Q}_2(i)/\mathbb{Q}_2) = 2$ is called ramification index

What is ramification?

- E/F extension of local fields
- e ramification index and f residue degree:

$$f = f(E/F) = [\kappa(E) : \kappa(F)]$$

- related by the formula $e \cdot f = [E : F]$
- The ramification index equals the group index of the value groups as subgroups of $\mathbb{R}$:

$$e(E/F) = \left(|E^\times|_E : |F^\times|_F\right)$$

What is ramification?

- E/F extension of local fields
- e ramification index and f residue degree:

$$f = f(E/F) = [\kappa(E) : \kappa(F)]$$

- related by the formula $e \cdot f = [E : F]$
- extension is called:
  - totally ramified, if $f = 1$
  - unramified, if $e = 1$
  - tamely ramified, if $\chi(F)$ does not divide e and otherwise wildly ramified.

What is ramification?

- $G_F = \mathrm{Gal}(\overline{F}/F)$ absolute galois group of local field F
- decreasing ramification filtration of G_F:

$$G_F \supset G_F^{(0)} \supset G_F^{(1)} \supset G_F^{(2)} \supset \dots$$

  where $G_F^{(0)} = I_F$ inertia subgroup and $G_F^{(1)} = P_F$ wild inertia subgroup
- $G_F^{\mathrm{ta}} = G_F/P_F$ admits explicit description
  $\Rightarrow$ canonical isomorphism $G_{\mathbb{Q}_p}^{\mathrm{ta}} \cong G_{\mathbb{F}_p((t))}^{\mathrm{ta}}$
  $\Rightarrow$ canonical assiciation of tame extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

What is ramification?

- Even better $n \geq 1$:

$$G_{\mathbb{Q}_p(p^{1/n})}/G^{(n)}_{\mathbb{Q}_p(p^{1/n})} \xrightarrow{\cong} G_{\mathbb{F}_p((t))(t^{1/n})}/G^{(n)}_{\mathbb{F}_p((t))(t^{1/n})}$$

- $n \to \infty$:

$$\mathbb{Q}_p(p^{1/n}) \text{ looks “almost” like } \mathbb{F}_p((t))(t^{1/n})$$

- Basis of tilting: from mixed to equal characteristic.
- Perfectoid Spaces $=$[1] *Framework for using equal characteristic results in the mixed characteristic world.*

[1]first approximation

Observations by Tate

- K finite extension of $\mathbb{Q}_p$
- Tower of extensions K_n/K:
  - K_n/K totally ramified
  - $\mathrm{Gal}(K_n/K) = (\mathbb{Z}/p^n\mathbb{Z})^h$ for some $h \geq 1$.
  - $K_\infty = \bigcup_{n\geq 1} K_n$
- Observation: If L/K_∞ finite extension
  - Ideal $(\mathrm{tr}_{L/K_\infty}(\mathcal{O}_L)) \subseteq \mathcal{O}_{K_\infty}$ contains $\mathfrak{p}_{K_\infty}$
  - Either equals $\mathfrak{p}_{K_\infty}$ or all of $\mathcal{O}_{K_\infty}$

Observations by Tate

- case of a finite extension E/F: trace Ideal related to different ideal of E/F
- measures ramification:

  bigger trace ideal $=$ less ramified extensions
- result: any finite extension of K_∞ is “almost” unramified
- hence: correspondig extension of $\mathcal{O}_{K_\infty}$ is “almost” étale

  (étale $=$ algebraic version of local diffeomorphism $=$ “algebraic unramified covering”)

Perfectoid fields

Definition

A Perfectoid field K is a complete non-archimedian field K of residue characteristic p, equipped with a non-discrete valuation of rank 1 ($|K^\times| \subseteq \mathbb{R}$ non-discrete), such that the Frobenius map

$$\mathcal{O}_K/(p) \longrightarrow \mathcal{O}_K/(p), \quad x \mapsto x^p$$

is surjective (every element has a p-th root).

- Example: completions of
  - $\mathbb{Q}_p(p^{1/p^\infty}) = \cup_{n\geq 1}\mathbb{Q}_p(p^{1/p^n})$
  - $\mathbb{F}_p((t))(t^{1/p^\infty}) = \cup_{n\geq 1}\mathbb{F}_p((t))(t^{1/p^n})$

Example

Show: completion of $\mathbb{Q}_p(p^{1/p^\infty})$ is a perfectoid field

- $\mathbb{Q}_p(p^{1/p^n})/\mathbb{Q}_p$ generated by $(X^{p^n} - p)$
- totally ramified with $e = p^n$
- $n \to \infty$: $|\mathbb{Q}_p(p^{1/p^\infty})^\times|$ non-discrete
- Frobenius

  $$\mathbb{Z}_p[p^{1/p^\infty}]/(p) \longrightarrow \mathbb{Z}_p[p^{1/p^\infty}]/(p)$$

  is surjective
- hence the corresponding completion is perfectoid

Almost étale coverings

- Let M be an $\mathcal{O}_K$-module
- M is almost zero if $\mathfrak{p}_K \cdot M = 0$
- localization functor: $M \mapsto M^a$

$$(\mathcal{O}_K - \mathrm{Mod}) \longrightarrow (\mathcal{O}_K^a - \mathrm{Mod}) = (\mathcal{O}_K - \mathrm{Mod})/(\text{almost zero})$$

(Serre quotient category) with right adjoint

$M \mapsto M_* = \mathrm{Hom}_{\mathcal{O}_K^a}(\mathcal{O}_K^a, M)$ functor of almost elements.

Almost étale coverings

- sequence of functors:

$$(\mathcal{O}_K - \mathrm{Mod}) \longrightarrow (\mathcal{O}_K^a - \mathrm{Mod}) \longrightarrow (K - \mathrm{Mod})$$

- geometric picture: composition corresponds to base change from integral structure to general fiber
- category in the middle: almost integral level, determined by the general fiber

Almost étale coverings

- $(\mathcal{O}_K^a - \mathrm{Mod})$ is an abelian tensor category
- notion of an $\mathcal{O}_K^a$-algebra as an algebra-object in $(\mathcal{O}_K^a - \mathrm{Mod})$ (A $\mathcal{O}_K^a$-Module with "multiplication" $\mu : A \otimes_{\mathcal{O}_K^a} A \longrightarrow A$)
- some commutative algebra:
  - flat, almost projective, almost finitely presented modules
  - unramified A-algebras
  - étale and finite étale A-algebras

Almost purity by Tate

Theorem (T)

Let L/K be a finite extension. Then $\mathcal{O}_L/\mathcal{O}_K$ is almost finite étale.

- Example: $p \neq 2$, $K_n = \mathbb{Q}_p(p^{1/p^n})$, $L_n = K_n(p^{1/2})$
  - $\mathcal{O}_{L_n} = \mathcal{O}_{K_n}[X]/(f)$ with $f = X^2 - p^{1/p^n}$
  - $p^{1/p^n} \in (f, f')\mathcal{O}_{K_n}[X]$
  - hence: up to p^{1/p^n}-torsion, $\mathcal{O}_{L_n}$ is étale over $\mathcal{O}_{K_n}$
  - $n \to \infty$: $\mathcal{O}_L$ almost étale over $\mathcal{O}_K$.
- general Philosophy:
  - Perfectoid fields are "deeply ramified" and absorb almost all ramification above them
  - hence: objects above them are almost unramified

Generalization by Scholze

Definition

A perfectoid K-algebra R is a Banach K-algebra, such that $R^\circ \subseteq R$ (subset of powerbound elements) is open and bounded, and the frobenius morphism

$$R^\circ/(\varpi) \longrightarrow R^\circ/(\varpi)$$

is surjective

Theorem (S)

Let S/R be finite étale. Then S is perfectoid and S° is uniformly almost finite étale over R°

How to see this?

- In the case of equal characteristic the theorems are "easy"
- for mixed characteristic, is there a way to switch to the equal characteristic setting and solve the problem there?
- answer: Yes, there is: tilting. (More about this tomorrow)

Why to do this?

- Almost purity can be translated into an assertion about cohomology groups
- for Tate: essential step in the proof of Hodge-Tate decomposition for p-divisible Groups:

Theorem (Hodge-Tate decomposition)

Let G be a p-divisible group. There is a canonical isomorphism of $G_{\mathbb{Q}_p}$-modules

$$\mathrm{Hom}(T(G), \mathbb{C}_p) \cong \mathfrak{t}_{G'}(\mathbb{C}_p) \oplus (\mathfrak{t}_G^*(\mathbb{C}_p) \otimes_{\mathbb{C}_p} \mathrm{Hom}(H, \mathbb{C}_p))$$

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