# Balanced triangulations on few vertices 

## Lorenzo Venturello

Algebra and Discrete Mathematics - Universität Osnabrück


19/02/2019

## Balanced simplicial complexes

## Definition

A d-dimensional simplicial complex on vertex set $[n]$ is balanced if there exists a map (which we call coloring) $\kappa:[n] \rightarrow[d+1]$ such that $\kappa(i) \neq \kappa(j)$ for every $\{i, j\} \in \Delta$.

Notable examples:

- Consider the $(d+1)$-dimensional cross-polytope $\mathcal{C}_{d+1}:=\operatorname{conv}\left(e_{1},-e_{1}, \ldots, e_{d+1},-e_{d+1}\right) \in \mathbb{R}^{d+1}$. Then $\partial \mathcal{C}_{d+1}$ is a balanced $d$-dimensional simplicial complex, with $\kappa\left(e_{i}\right)=\kappa\left(-e_{i}\right)=i$.

- The barycentric subdivision of any simplicial complex is balanced. E.g.,



## Interesting geometric realizations: spheres

We describe four families of simplicial complexes.

- A combinatorial $d$-sphere is a $d$-dimensional simplicial complex that has a common subdivision with the boundary of the $(d+1)$-simplex.


## in

- A simplicial $d$-sphere is a $d$-dimensional simplicial complex $\Delta$ s.t. $|\Delta| \cong S^{d}$.


## In

- A $K$-homology $d$-sphere is a $d$-dimensional simplicial complex $\Delta$ such that $\widetilde{H}_{i}\left(\mathrm{k}_{\Delta}(F) ; K\right) \cong \widetilde{H}_{i}\left(S^{d-\operatorname{dim}(F)-1} ; K\right)$ for a fixed field $K$ and for every $i$.


## Interesting geometric realizations: manifolds

We describe three families of simplicial complexes.

- A combinatorial $d$-manifold is a $d$-dimensional simplicial complex s.t. every vertex link is a combinatorial ( $d-1$ )-sphere.


## In

- A simplicial $d$-manifold is a $d$-dimensional simplicial complex $\Delta$ s.t. $|\Delta| \cong M$, where $M$ is a (triangulable) topological manifold.

In

- A $K$-homology $d$-manifold is a $d$-dimensional simplicial complex $\Delta$ such that $\widetilde{H}_{i}\left(\mathrm{Ik}_{\Delta}(F) ; K\right) \cong \widetilde{H}_{i}\left(S^{d-\operatorname{dim}(F)-1} ; K\right)$ for a fixed field $K$, for every $\varnothing \neq F \in \Delta$ and for every $i$.


## Small balanced triangulations

- Applying barycentric subdivision to the standard (i.e., non-balanced) minimal triangulations yields complexes with a large number of vertices.
- $\partial \mathcal{C}_{d+1}$ is the balanced vertex minimal triangulation of $S^{d}$.
- Klee and Novik (2014) constructed triangulations of $S^{1} \times S^{d-1}$ (on $3(d+1)$ vertices if $d$ is even and on $3 d+5$ if $d$ is odd) and $S^{1} \times S^{d-1}$ (on $3(d+1)$ vertices if $d$ is odd and on $3 d+5$ if $d$ is even). The proof was completed by Zheng.
- Wang and Zheng (2018) constructed a balanced triangulation of $S^{2} \times S^{d-2}$ on $4 d$ vertices.


## Main Problem

Find balanced combinatorial triangulations of a given manifold on few (possibly the minimum number of) vertices.

Idea: Find a finite set of operations that preserve the PL-homeomorphism type.

## Bistellar flips

## Definition

Let $\Delta$ be a $d$-dimensional simplicial complex, and let $\Gamma$ be an induced subcomplex of $\Delta$ that is:

- PL-homeomorphic to a $d$-ball,
- Isomorphic to a subcomplex of $\partial \Delta_{d+1}$ (the boundary of the $(d+1)$-simplex).

A bistellar flip on $\Delta$ replaces all the facets in $\Gamma$ with those in $\partial \Delta_{d+1} \backslash \Gamma$.


## Bistellar flips

## Definition

Let $\Delta$ be a $d$-dimensional simplicial complex, and let $\Gamma$ be an induced subcomplex of $\Delta$ that is:

- PL-homeomorphic to a d-ball,
- Isomorphic to a subcomplex of $\partial \Delta_{d+1}$ (the boundary of the $(d+1)$-simplex).

A bistellar flip on $\Delta$ replaces all the facets in $\Gamma$ with those in $\partial \Delta_{d+1} \backslash \Gamma$.

## Theorem (Pachner)

Two combinatorial manifolds are PL-homeomorphic if and only if they are related by a sequence of bistellar flips.

Lutz wrote a computer program called BISTELLAR to search through the set of combinatorial triangulations of a given manifold.

## Cross-flips

Bistellar flips do not preserve balancedness.

## Definition (Izmestiev-Klee-Novik)

Let $\Delta$ be a balanced $d$-dimensional simplicial complex, and let $\Gamma$ be an induced subcomplex of $\Delta$ that is:

- PL-homeomorphic to a $d$-ball
- Isomorphic to a subcomplex of $\partial \mathcal{C}_{d+1}$

A cross flip on $\Delta$ replaces all the facets in $\Gamma$ with those in $\partial \mathcal{C}_{d+1} \backslash \Gamma$.


## Cross-flips

Bistellar flips do not preserve balancedness.

## Definition (Izmestiev-Klee-Novik)

Let $\Delta$ be a balanced $d$-dimensional simplicial complex, and let $\Gamma$ be an induced subcomplex of $\Delta$ that is:

- PL-homeomorphic to a d-ball
- Isomorphic to a subcomplex of $\partial \mathcal{C}_{d+1}$

A cross flip on $\Delta$ replaces all the facets in $\Gamma$ with those in $\partial \mathcal{C}_{d+1} \backslash \Gamma$.

## Theorem (Izmestiev-Klee-Novik)

Two balanced combinatorial manifolds are PL-homeomorphic if and only if they are related by a sequence of basic cross flips.

## Theorem (Juhnke-Kubitzke, V.)

In dimension d there are precisely $2^{d+1}-2$ non-isomorphic non-trivial basic cross-flips. Moreover $2^{d}-1$ of these flips suffice to connect any two PL-homeomorphic balanced combinatorial d-manifolds.

## The implementation

Goal: Obtain small balanced triangulations of surfaces and 3-manifolds.
Starting point: The barycentric subdivision of a simplicial complex $\Delta$ is balanced triangulation of $|\operatorname{Bd}(\Delta)| \cong|\Delta|$.

## Problem

Given a balanced combinatorial $d$-manifold $\Delta$ how do I detect all applicable cross-flips?
Proposed solution: fix a flip $\Gamma \rightarrow \partial \mathcal{C}_{d+1} \backslash \Gamma$.

- Consider the dual graph $\mathcal{G}(\Delta)$, given by

$$
V(\mathcal{G}(\Delta)):=\{F \in \Delta: \operatorname{dim}(F)=d\}, \quad E(\mathcal{G}(\Delta)):=\left\{\left\{F_{i}, F_{j}\right\}: \operatorname{dim}\left(F_{i} \cap F_{j}\right)=d-1\right\} .
$$

- List all subgraphs of $\mathcal{G}(\Delta)$ isomorphic to $\mathcal{G}(\Gamma)$. We can use efficient algorithms for this task (e.g., VF2).
- Check if the subcomplex corresponding to $\mathcal{G}(\Gamma)$ is an induced subcomplex isomorphic to $\Gamma$.


$$
\mathcal{G}(\Delta)=
$$





Does not correspond to an induced subcomplex.

## The implementation

## Problem

Given the list of all applicable cross-flips, which one do we choose?
Proposed heuristic solution: Let

$$
\chi_{\Gamma}(\Delta):=(\Delta \backslash \Gamma) \cup \partial \mathcal{C}_{d+1} \backslash \Gamma .
$$

Choose the flip $\Gamma \rightarrow \mathcal{C}_{d+1} \backslash \Gamma$ among those which:

- maximize $\left|\left\{v \in \chi_{\Gamma}(\Delta): \operatorname{deg}(v)=2 d\right\}\right|$.

Fact: If a vertex $v$ can be removed by a flip then $\operatorname{deg}(v)=2 d$.

- maximize $\sum_{v \in \chi_{\Gamma}(\Delta), \operatorname{dim}(v)=0} \operatorname{deg}(v)^{2}$.

This forces the connectivity of the vertices to be inhomogeneous.

## Results in dimension 2

## Theorem (V.)

The vertex-minimal balanced triangulation of $\mathbb{R} \mathbf{P}^{2}$ has 9 vertices.


The non-balanced vertex minimal triangulation has 6 vertices.
The balanced vertex-minimality follows from a result of Klee and Novik: if a combinatorial manifold is not an homology sphere then it contains at least three vertices per color class.

## Results in dimension 2

The dunce hat is a topological space obtained as a quotient of the 2-disk according to the following orientation:


Properties:

- Cohen-Macaulay;
- Partitionable;
- Non-constructible, hence non-shellable;
- Contractible;
- Non-collapsible;

Low dimensional pathological example.

## Results in dimension 2

## Theorem (V.)

The vertex-minimal balanced triangulation of the dunce hat has 11 vertices.


The non-balanced vertex minimal triangulation has 8 vertices.
The proof of balanced vertex-minimality is more technical, and divided in several cases.

## Results in dimension 2

| $\|\Delta\|$ | $\operatorname{Min} f(\Delta)$ | $f(\operatorname{Bd}(\Delta))$ | Min. Bal. $f$ known | Notes |
| :--- | :--- | :--- | :--- | :--- |
| $S^{2}$ | $(1,4,6,4)$ | $(1,14,36,24)$ | $(1,6,12,8)^{*}$ | $\partial \mathcal{C}_{3}$ |
| $\mathbb{T}$ | $(1,7,21,14)$ | $(1,42,126,84)$ | $(1,9,24,16)^{*}$ | Klee-Novik |
| $\mathbb{T}^{\# 2}$ | $(1,10,36,24)$ | $(1,70,216,144)$ | $(1,12,42,28)$ |  |
| $\mathbb{T}^{\# 3}$ | $(1,10,42,28)$ | $(1,80,252,168)$ | $(1,14,54,36)$ |  |
| $\mathbb{T}^{\# 4}$ | $(1,11,51,34)$ | $(1,96,306,204)$ | $(1,14,60,36)$ |  |
| $\mathbb{T}^{\# 5}$ | $(1,12,60,40)$ | $(1,112,360,240)$ | $(1,16,72,48)$ |  |
| $\mathbb{R} \mathbf{P}^{2}$ | $(1,6,15,10)$ | $(1,31,90,60)$ | $(1,9,24,16)^{*}$ | $\Delta_{9}^{\mathbb{R} \mathbf{R}^{2}}$ |
| $\left(\mathbb{R} \mathbf{P}^{2}\right)^{\# 2}$ | $(1,8,24,16)$ | $(1,48,144,96)$ | $(1,11,33,22)^{*}$ | Klee-Novik |
| $\left(\mathbb{R} \mathbf{P}^{2}\right)^{\# 3}$ | $(1,9,30,20)$ | $(1,59,180,120)$ | $(1,12,39,26)$ |  |
| $\left(\mathbb{R} \mathbf{P}^{2}\right)^{\# 4}$ | $(1,9,33,22)$ | $(1,64,198,132)$ | $(1,12,42,28)$ |  |
| $\left(\mathbb{R} \mathbf{P}^{2}\right)^{\# 5}$ | $(1,9,36,24)$ | $(1,69,216,144)$ | $(1,13,48,32)$ |  |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  |

## Results in dimension 3

## Theorem (V.)

The vertex-minimal balanced triangulations of $\mathbb{R} \mathbf{P}^{3}$ have 16 vertices.
Zheng proved that any balanced triangulation of a manifold $M$ with $H_{1}(M ; \mathbb{Z}) \cong \mathbb{Z}_{n}$ has at least four vertices per color class.
The balanced triangulation on 16 vertices is not unique.
In particular there exists a balanced triangulation of $\mathbb{R} \mathbf{P}^{3}$ on 16 vertices that is centrally symmetric and has all vertex links isomorphic to the following 2 -sphere.


## Results in dimension 3

## Theorem (V.)

The vertex-minimal balanced triangulations of $\mathbb{R} \mathbf{P}^{3}$ have 16 vertices.
Zheng proved that any balanced triangulation of a manifold $M$ with $H_{1}(M ; \mathbb{Z}) \cong \mathbb{Z}_{n}$ has at least four vertices per color class.
The balanced triangulation on 16 vertices is not unique.
In particular there exists a balanced triangulation of $\mathbb{R} \mathbf{P}^{3}$ on 16 vertices that is centrally symmetric and has all vertex links isomorphic to the following 2 -sphere.


## Conjecture (by the engineer's induction)

The balanced vertex-minimal triangulations of $\mathbb{R} \mathbf{P}^{d}$ have $(d+1)^{2}$ vertices.

## Results in dimension 3

| $\|\Delta\|$ | Min $f(\Delta)$ | Min. Bal. $f$ known | Notes |
| :--- | :--- | :--- | :--- |
| $S^{3}$ | $(1,5,10,10,5)$ | $(1, \mathbf{8}, 24,32,16)^{*}$ | $\partial \mathcal{C}_{4}$ |
| $S^{2} \times S^{1}$ | $(1,10,42,64,32)$ | $(1, \mathbf{1 4}, 64,100,50)^{*}$ | Klee-Novik |
| $S^{2} \times S^{1}$ | $(1,9,36,54,27)$ | $(1,12,54,84,42)^{*}$ | Klee-Novik |
| $\mathbb{R} \mathbf{P}^{3}$ | $(1,11,51,80,40)$ | $(1,16,88,144,72)^{*}$ | $\Delta_{16}^{\mathbb{R} P^{3}}$ |
| $L(3,1)$ | $(1,12,66,108,54)$ | $(1, \mathbf{1 6}, 96,160,80)^{*}$ | Zheng |
| $L(4,1)$ | $(1,14,84,140,70)$ | $(1,20,132,224,112)$ |  |
| $L(5,1)$ | $(1,15,97,164,82)$ | $(1, \mathbf{2 2}, 152,260,130)$ |  |
| $L(5,2)$ | $(1,14,86,144,72)$ | $(1,20,132,224,112)$ |  |
| $L(6,1)$ | $(1,16,110,188,94)$ | $(1, \mathbf{2 4}, 176,304,152)$ |  |
| $\left(S^{2} \times S^{1}\right)^{\# 2}$ | $(1,12,58,92,46)$ | $(1,16,84,136,68)$ |  |
| $\left(S^{2} \times S^{1}\right)^{\# 2}$ | $(1,12,58,92,46)$ | $(1,16,84,136,68)$ |  |
| $\left(S^{2} \times S^{1}\right) \# \mathbb{R} \mathbf{P}^{3}$ | $(1,14,73,118,59)$ | $(1, \mathbf{2 0}, 118,196,98)$ |  |
| $\left(\mathbb{R} \mathbf{P}^{3}\right)^{\# 2}$ | $(1,15,86,142,71)$ | $(1, \mathbf{2 1}, 137,232,116)$ |  |
| $\left(S^{2} \times S^{1}\right)^{\# 3}$ | $(1,13,72,118,59)$ | $(1, \mathbf{2 0}, 118,196,98)$ |  |
| $\left(S^{2} \times S^{1}\right)^{\# 3}$ | $(1,13,72,118,59)$ | $(1, \mathbf{1 9}, 111,184,92)$ |  |
| $S^{1} \times S^{1} \times S^{1}$ | $(1,15,105,180,90)$ | $(1,24,168,288,144)$ |  |
| Oct. space | $(1,15,102,174,87)$ | $(1,24,168,288,144)$ |  |
| Cube space | $(1,15,90,150,75)$ | $(1,23,157,268,134)$ |  |
| Poincaré | $(1,16,106,180,90)$ | $(1, \mathbf{2 6}, 180,308,154)$ |  |
| $\mathbb{R} \mathbf{P}^{2} \times S^{1}$ | $(1,14,84,140,70)$ | $(1, \mathbf{2 4}, 156,264,132)$ |  |

## Non-combinatorial spheres

## Theorem (V.)

There exists a balanced triangulation of Poincaré homology 3-sphere, with f-vector (1, 26, 180, 308, 154).

A result of Cannon and Edwards states that the double suspension of an homology 3 -sphere $\Delta$ is homeomorphic to $S^{5}$. Still $\Delta$ will appear as one of the links, which is an obstruction to being combinatorial.

## Corollary

There exist balanced non-combinatorial $d$-spheres on $2 d+20$ vertices, for every $d \geq 5$.

## Two triangulations of $S^{3}$

## Definition

A pure $d$-dimensional simplicial complex is shellable if there exists an ordering $F_{1}, \ldots, F_{m}$ of its facets such that the complex $\left\langle F_{1}, \ldots, F_{i-1}\right\rangle \cap\left\langle F_{i}\right\rangle$ is pure and ( $d-1$ )-dimensional for every $1 \leq i \leq m$.

- Boundary complexes of simplicial polytopes are shellable.
- It is interesting and challenging to find balanced non-shellable triangulations of spheres.
- Barycentric subdivision typically turns non-shellable combinatorial sphere into shellable ones (if allow iterated subdivisions the "typically" becomes "always" by a result of Adiprasito and Benedetti).


## Theorem (V.)

There exists a balanced non-shellable combinatorial 3 -sphere with $f$-vector (1, 28, 204, 352, 176).


Figure: 3-ball containing a triple-trefoil knot. (Benedetti and Ziegler, 2011).

- There is a triangulation of the 3 -sphere containing a triple-trefoil knot on three edges, with the $f$-vector $(1,18,143,250,125)$ (Benedetti-Lutz). By a result of Lickorish it is not shellable.
- Ehrenborg and Hachimori proved that the barycentric subdivision of a 3-sphere with such a knot is not shellable.
- We run our computer program applying only flips that do not untie the knot.
- Starting from a 3 -sphere containing a double trefoil-knot we obtain a balanced triangulation of $S^{3}$ that is shellable but not vertex-decomposable.


## Thanks for your attention

