# Conditional Independence Ideals with Hidden Variables 

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February, 2019

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## Overview

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## Guiding Example

## Question

Does the temperature affect the price of fruit?
Define two random variables,

$$
F \in\left\{\begin{array}{c}
\text { 'cheap', } \\
\text { 'expensive' }
\end{array}\right\} \text { fruit price, } T \in\left\{\begin{array}{c}
\text { 'hot', } \\
\text { 'cold' }
\end{array}\right\} \text { temperature. }
$$

Suppose we observe these probabilities:

| $P^{F, T}$ | 'cheap' | 'expensive' |
| :---: | :---: | :---: |
| 'cold' | 0.1 | 0.4 |
| 'hot' | 0.2 | 0.3 |

Let's check if $F$ and $T$ are dependent,

| $P^{F, T}$ | 'cheap' | 'expensive' |
| :---: | :---: | :---: |
| 'cold' | 0.1 | 0.4 |
| 'hot' | 0.2 | 0.3 |

$$
\mathbb{P}(F=\text { 'cheap' } \mid T=\text { 'hot' })=0.4 \neq 0.3=\mathbb{P}(F=\text { 'cheap' }) \text {. }
$$

So $F$ and $T$ are dependent.

## Remark

The variables $F$ and $T$ are independent if and only if $P^{F, T}$ has rank 1 . In such a case we write $F \Perp T$.

Hidden explanation: 'Temperature' $\rightarrow$ 'Availability' $\rightarrow$ 'Fruit Price'. Consider,

$$
A \in\left\{\begin{array}{c}
\text { 'few' } \\
\text { 'many' }
\end{array}\right\} \text { availability of fruit. }
$$

| 'many' | 'cheap' | 'expensive' | 'few' | 'cheap' | 'expensive' |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 'cold' | 0.1 | 0.1 | 'cold' | 0 | 0.3 |
| 'hot' | 0.2 | 0.2 | 'hot' | 0 | 0.1 |

So we say Price $(F)$ is independent of Temperature $(T)$ given Availability $(A)$ and write $F \Perp T \mid A$.

## Cl (Conditional Independence) Statements

We write $X \Perp Y \mid Z$ (or more generally $X_{A} \Perp X_{B} \mid X_{C}$ ) if

$$
\mathbb{P}(X=x, Y=y \mid Z=z)=\mathbb{P}(X=x \mid Z=z) \mathbb{P}(Y=y \mid Z=z)
$$

for any $x, y, z$, or equivalently,

$$
p_{x y z} p_{x^{\prime} y^{\prime} z}-p_{x y^{\prime} z} p_{x^{\prime} y z}=0, \text { for any } x, x^{\prime}, y, y^{\prime}, z
$$

where $p_{x y z}=\mathbb{P}(X=x, Y=y, Z=z)$.

## CI Questions

- Let $X_{1}, \ldots, X_{n}$ be finite random variables on $\mathcal{X}_{1}, \ldots, \mathcal{X}_{n}$.
- Let $\mathcal{C}$ be a collection of independence statements.

What can we say about the probability distributions that satisfy $\mathcal{C}$ ?

For example, let $\mathcal{C}=\{X \Perp Y|Z, X \Perp Z| Y\}$. The intersection axiom says, if the joint distribution has full support then $X \Perp\{Y, Z\}$.

## CI Ideals

- Let $R=\mathbb{C}\left[p_{x_{1}, \ldots, x_{n}}\right]$ be the polynomial ring with one variable for each element of $\mathcal{X}_{1} \times \cdots \times \mathcal{X}_{n}$


## Definition

Let $I_{\mathcal{C}}$ be the $\mathbf{C l}$ ideal generated by all 'rank 1 ' conditions of the form,

$$
p_{x y z} p_{x^{\prime} y^{\prime} z}-p_{x y^{\prime} z} p_{x^{\prime} y z} .
$$

Where $x \in \mathcal{X}_{A}, y \in \mathcal{X}_{B}, z \in \mathcal{X}_{C}$ for each $X_{A} \Perp X_{B} \mid X_{C} \in \mathcal{C}$.

For example, if $\mathcal{C}=\{X \Perp Y|Z, X \Perp Z| Y\}$ then by considering its primary decomposition,

$$
I_{\mathcal{C}} \subseteq I_{X \Perp\{Y, Z\}}
$$

## Hidden Variables

## Observation

Suppose $X \Perp Y \mid Z$ then for each $z, P_{z}=\left(p_{x y z}\right)_{x y}$ has rank 1 and so,

$$
P^{x, Y}=\left(\sum_{z} p_{x y z}\right)_{x y} \text { has rank at most }|\mathcal{Z}| \text {. }
$$

- Let $X_{1}, \ldots, X_{n}$ [observed] and $H_{1}, \ldots, H_{m}$ [hidden] be finite random variables on $\mathcal{X}_{1}, \ldots, \mathcal{X}_{n}, \mathcal{H}_{1}, \ldots, \mathcal{H}_{m}$.
- Let $\mathcal{C}$ be a collection of independence statements.
- Let $R=\mathbb{C}\left[p_{x_{1}, \ldots, x_{n}}\right]$ be the polynomial ring with variables with one variable for each outcome of the observed variables.

The $\mathbf{C I}$ Ideal $I_{\mathcal{C}} \subseteq R$ is generated by all 'rank 1' conditions.

## Our Example

- $X \in[d], Y_{1} \in[k], Y_{2} \in[\ell], H \in[\ell-1]$ random variables.
- $\mathcal{C}=\left\{X \Perp Y_{1}\left|Y_{2}, X \Perp Y_{2}\right|\left\{Y_{1}, H\right\}\right\} \mathrm{Cl}$ statements.
- $R=\mathbb{C}\left[p_{1,(1,1)}, \ldots, p_{d,(k, \ell)}\right]$ polynomial ring.
- $P=\left(p_{i j}\right)$ is a $d \times k \ell$ matrix in the variables.

$$
P=\left[\begin{array}{cccccc}
p_{1,(1,1)} & \cdots & p_{1,(k, 1)} & p_{1,(1,2)} & \cdots & p_{1,(k, \ell)} \\
\vdots & & \vdots & \vdots & & \vdots \\
p_{d,(1,1)} & \cdots & p_{d,(k, 1)} & p_{d,(1,2)} & \cdots & p_{d,(k, \ell)}
\end{array}\right]
$$

- $I_{\mathcal{C}}$ is generated by certain minors of $P$.


## Our Ideal

Write $\mathcal{Y}=\mathcal{Y}_{1} \times \mathcal{Y}_{2}=[k] \times[\ell]$ as,

$$
\mathcal{Y}=\left[\begin{array}{cccc}
1 & k+1 & \ldots & (\ell-1) k+1 \\
2 & k+2 & \ldots & (\ell-1) k+2 \\
\vdots & \vdots & \ddots & \vdots \\
k & 2 k & \ldots & \ell k
\end{array}\right] .
$$

The ideal $I_{\mathcal{C}}$ is generated by minors of the form $[R \mid C]_{P}$ where

- $C$ is a 2 -subset of a column of $\mathcal{Y}$ or
- $C$ is a row of $\mathcal{Y}$,
- $R$ is a $|C|$-subset of $[d]$.


## Results

Let $\mathcal{L}=\left\{\left\{\left(i_{1}, j_{1}\right), \ldots,\left(i_{k}, j_{k}\right)\right\}:\left|\left\{i_{1}, \ldots i_{k}\right\}\right|=k,\left|\left\{j_{1}, \ldots, j_{k}\right\}\right| \geq 2\right\}$.
For each $S \in \mathcal{L}$ let $I_{S}=I_{\mathcal{C}}+\left\langle x_{i, j}: i \in\{1, \ldots, d\}, j \in S\right\rangle$.
Let $I_{0}=I_{\mathcal{C}}+\langle$ all $\ell$-minors of $P\rangle$.

## Theorem

- The generating sets of $I_{0}$ and $I_{S}$ form Gröbner bases.
- The ideals $I_{0}$ and $I_{S}$ are prime.
- The radical of $I_{\mathcal{C}}$ has minimal primary decomposition,

$$
\sqrt{I_{C}}=I_{0} \cap \bigcap_{S \in \mathcal{L}} I_{S}
$$

Note that,

$$
I_{0}=I_{X \Perp Y_{1}\left|Y_{2}, X \Perp\{Y 1, Y 2\}\right| H}
$$

is the only prime component that contains no variables. Any distribution with full support lies in $V\left(I_{0}\right)$

## Open Questions

- What happens in the case $H \in[t-1]$ with $t<\ell$ ? (Taking $t$ minors of the rows of $\mathcal{Y}$.)
- For $d=3, k=2, \ell=4, t=3$, calculation in Macaulay2 shows there are 43 prime components all of which are determinantal ideals.
- What if $\mathcal{C}=\left\{X \Perp Y_{1}\left|Y_{2} H_{1}, X \Perp Y_{2}\right| Y_{1} H_{2}\right\}$ ? (Taking larger minors of the columns of $\mathcal{Y}$.)
- For $d=k=3, \ell=4$ and $\left|\mathcal{H}_{1}\right|=\left|\mathcal{H}_{2}\right|=2$, calculation by Pfister and Steenpass ${ }^{1}$ shows there are two prime components $I_{0}$ and $I_{1}$, neither contain variables and one is not determinantal.


## Notes on calculations

Details about the ideals $I_{0}$ and $I_{1}$.
$>$ The ideal $I_{0}$ is generated by all 3 -minors of $P$.
$>$ The ideal $I_{1}$ is generated by 44 polynomials; the first 16 are the generators of $I_{\mathcal{C}}$.
$>$ The remaining generators $G^{\prime}$ are homogeneous of degree 12 and for each generator, every term contains a variable $p_{1, i}, p_{2, i}$ or $p_{3, i}$ for each index $i \in[12]$.
$>$ For each generator in $G^{\prime}$, every term has the same number of $p_{1, i}, p_{2, i}$ and $p_{3, i}$.

