Tope Arrangements and Determinantal Varieties

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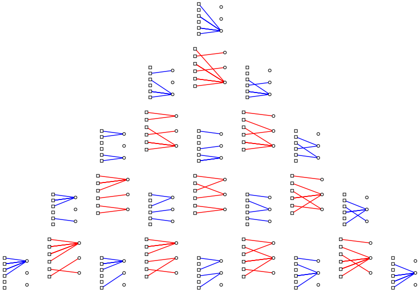

Graduate Student Meeting on Applied Algebra and Combinatorics
Joint with Georg Loho, LSE

Tope Arrangements

What is a tope arrangement?

Consider the bipartite node sets $[n] \sqcup [d]$.

Definition

A *tope* is a bipartite graph whose left nodes $[n]$ all have degree one.

Let $P_{k,d} = k\Delta_{d-1} \cap \mathbb{Z}^d$ be the of lattice points $(d-1)$-simplex scaled by k.

Definition

An *(n, d)-tope arrangement* is a collection of topes on $[n] \sqcup [d]$ such that:

- the right degree vectors are in bijection with $P_{n-d,d}$, the lattice points of $(n-d)\Delta_{d-1}$.
- if two topes contain a matching on a subset of nodes $J \sqcup I$, it is the same matching.

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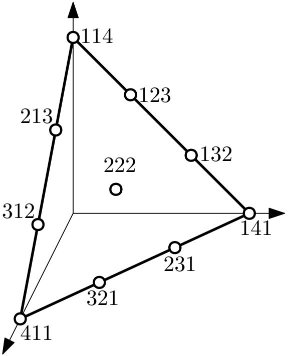
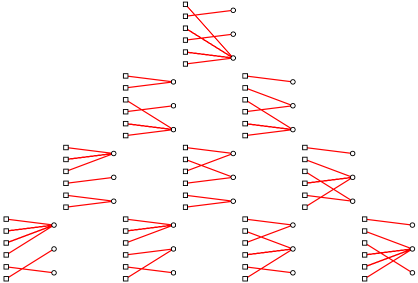


Lattice points of $3\Delta_2$



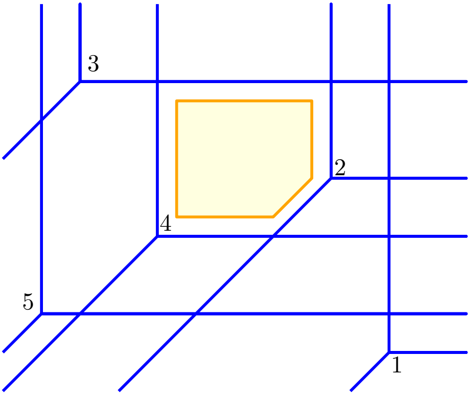
$(6, 3)$-tope arrangement

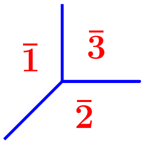
Example 1: Tropical Hyperplane Arrangements

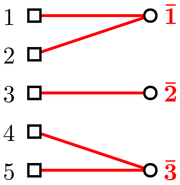
Where do tope arrangements naturally occur?

A *tropical hyperplane* is a fan in $\mathbb{R}^{d-1}$ with d maximal cones, labelled by $\{1, \dots, d\}$.
An arrangement of n tropical hyperplanes decomposes $\mathbb{R}^{d-1}$ into regions.
Each region has a corresponding bipartite graph on $[n] \sqcup [d]$ with edges

$$(j, i) \;\Leftrightarrow\; \text{the region is in the } i\text{-th cone of hyperplane } j$$



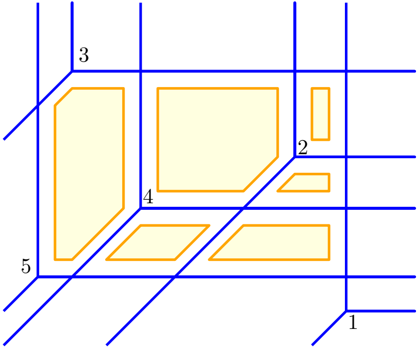


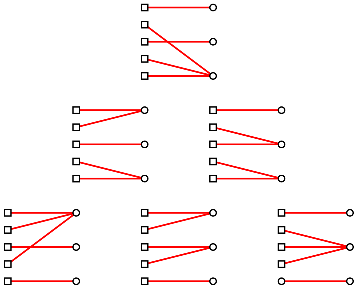


Example 1: Tropical Hyperplane Arrangements

Proposition (Ardila, Develin, Sturmfels)

The bipartite graphs from the bounded regions of an arrangement of n tropical hyperplanes in $\mathbb{R}^{d-1}$ form an (n, d)-tope arrangement.





Tropical Hyperplane Arrangement $\subset$ Tropical Oriented Matroid $\subset$ Tope Arrangement

Example 2: Determinantal Varieties

Where do tope arrangements occur classically?

Let $\nabla_{d,n}$ be the variety of degenerate $(d \times n)$-matrices

$$\nabla_{d,n} = \left\{ X \in \mathbb{C}^{d\times n} \;\middle|\; \mathrm{rk}(X) < d \right\} \ .$$

It is cut out by the ideal

$$I_{d,n} = \langle \det(M|_J) \mid J \subset [n] \ , \ |J| = d \rangle \subset \mathbb{C}[x_{ij}] \ , \ M = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dn} \end{bmatrix}$$

generated by the maximal minors of the $(d \times n)$-matrix of indeterminates.

Example

$\nabla_{2,3} \subset \mathbb{C}^6$ is the variety cut out by the ideal

$$I_{2,3} = \langle x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22} \rangle \subset \mathbb{C}[x_{11}, \dots, x_{23}] \ ,$$

the ideal generated by the maximal minors of the matrix $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$.

Example 2: Determinantal Varieties

Definition

Let $W = (w_{ij}) \in \mathbb{R}^{d \times n}$ be a (generic) matrix of weights. The *weight* of a monomial $x^{\mathbf{a}}$ is

$$\sum_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}} w_{ij} a_{ij} \quad .$$

The *initial form* $in_W(f)$ of a polynomial f w.r.t W is the monomial of least weight.

Term orderings and initial forms are the main tools of Gröbner bases.

Example

$$I_{2,3} = \langle x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{23} - x_{13}x_{22} \rangle \subset \mathbb{C}[x_{11}, \ldots, x_{23}]$$

Let $W = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$. The initial forms of each of the generators of $I_{2,3}$ are

$$in_W(x_{11}x_{22} - x_{12}x_{21}) = x_{11}x_{22}$$
$$in_W(x_{11}x_{23} - x_{13}x_{21}) = x_{11}x_{23}$$
$$in_W(x_{12}x_{23} - x_{13}x_{22}) = x_{12}x_{23}$$

Example 2: Determinantal Varieties

$$M = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dn} \end{bmatrix} \text{ with term order induced by } W.$$

Definition

A *tope monomial* is a set of n variables from M such that

- There is exactly one variable from each column of M.
- For any subset with exactly one variable from each row of M, the product of those variables is the initial form of a maximal minor of M.

Theorem (Sturmfels, Zelevinksy / Loho, S)

The indices of a tope monomial form a tope. The set of all tope monomials w.r.t a weight matrix form a tope arrangement.

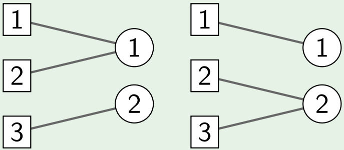
Example 2: Determinantal Varieties

Example

Consider the previous example, the initial forms were $\{x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23}\}$. There are precisely two tope monomials:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

The corresponding tope arrangement is in bijection with the lattice points of Δ_1.



$x_{11}x_{12}x_{23}$ $\quad$ $x_{11}x_{22}x_{23}$

This was not the language Sturmfels and Zelevinksy used. They instead observed that the initial forms of a maximal minor induce a matching on $[n] \sqcup [d]$. The set of all these define a *matching field*.

Matching Fields

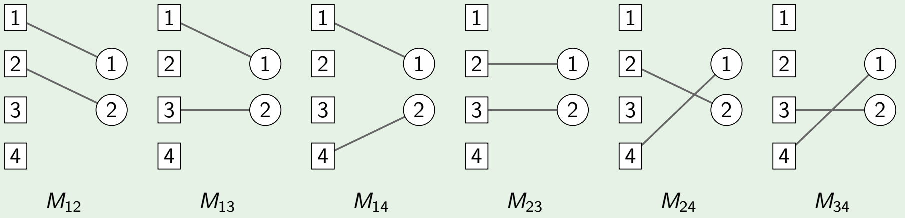
What is a matching field?

Definition

A *matching field* $\mathcal{M} = (m_J)$ on $[n] \sqcup [d]$ is a collection of matchings on $J \sqcup [d]$, one for each d-subset $J \subset [n]$.

- It is *coherent* if induced by a weight matrix.
- It is *linkage* if for each m_J and $k \in [n] \setminus J$, there exists $j \in J$ such that m_J and $m_{J \setminus j \cup k}$ differ by a flip (basis exchange axiom).

Example ($n = 4, d = 2$)

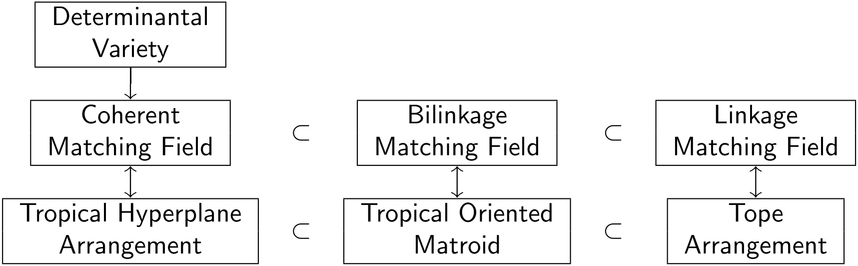


Correspondence with Matching Fields

How are matching fields and tope arrangements related?

Theorem (Loho, S '18)

Tope arrangements and linkage matching fields are cryptomorphic.



Chow Graphs

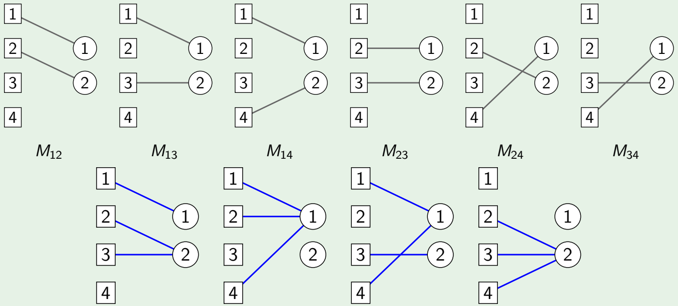
What have tope arrangements got over matching fields?

Sturmfels and Zelevinksy were interested in the Chow polytope $\mathrm{Ch}(\nabla_{\mathrm{d,n}})$ of $\nabla_{d,n}$. In studying this, they considered the following graphs:

Definition

Fix a matching field $\mathcal{M} = (m_J)$. A *Chow graph* Ω is a minimal bipartite graph such that $\Omega \cap m_J \neq \emptyset$ for all m_J.

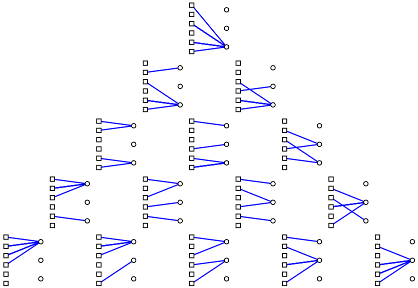
Example



Chow Conjecture

Conjecture (Sturmfels, Zelevinsky '93)

- *The Chow graphs of a linkage matching field are in bijection with $P_{n-d+1,d}$, the lattice points of $(n-d+1)\Delta_{d-1}$ via their right degree vector.*
- *The Chow graphs determine the linkage matching field.*

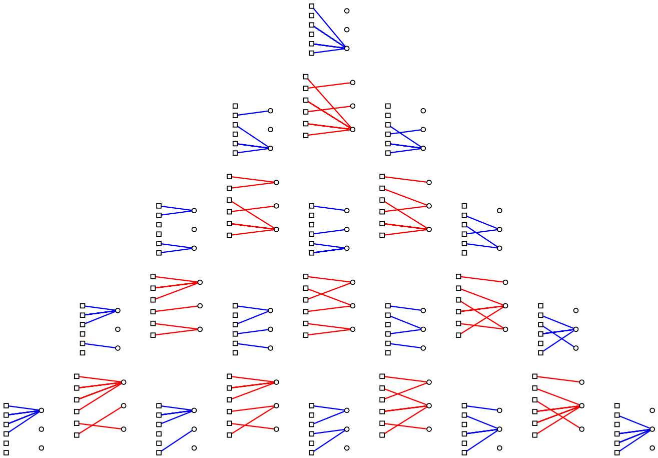


- Bernstein, Zelevinsky '93 - holds for coherent matching fields.
- Loho, S '18 - holds for all linkage matching fields.

Chow Conjecture

Theorem (Loho, S '18)

The Chow graphs of $\mathcal{M}$ can be recovered from the associated tope arrangement via intersections. This induces the bijection with $P_{n-d+1,d}$. Furthermore, they determine the tope arrangement via unions.



Final Thoughts

Tope Arrangements $\longleftrightarrow$ Matching Fields $\longleftrightarrow$ Chow Graphs

Question

Can one formulate a characterisation of Chow graphs that doesn't depend on the matching field or tope arrangement?

References: Matching fields and lattice points of simplices, Georg Loho and Ben Smith, arXiv:1804.01595, (2018)

Maximal minors and their leading terms, Bernd Sturmfels and Andrei Zelevinsky, Advances in Mathematics, **98** (1993), 65–112

Combinatorics of maximal minors, David Bernstein and Andrei Zelevinsky, Journal of Algebraic Combinatorics, **2** (1993), 111–121