

Lamprini Ananiadi

Gröbner Bases for toric staged trees

(joint work with Eliana Duarte)

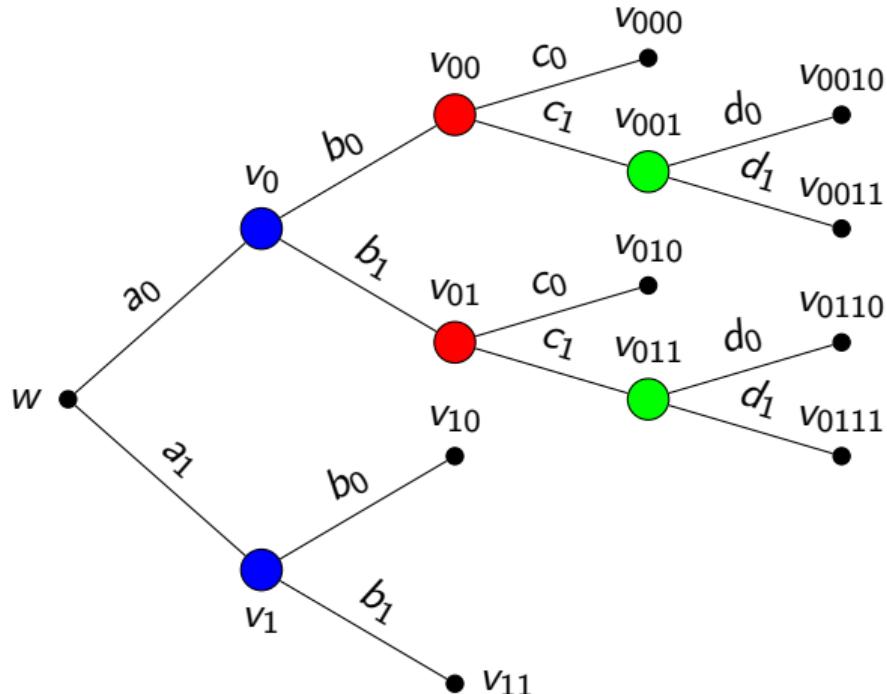
Leipzig, February 18, 2019

Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg



DFG-Graduiertenkolleg
**MATHEMATISCHE
KOMPLEXITÄTSREDUKTION**

Staged trees



$$\theta_T = (a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1)$$

$$a_0 + a_1 = 1$$

$$b_0 + b_1 = 1$$

$$c_0 + c_1 = 1$$

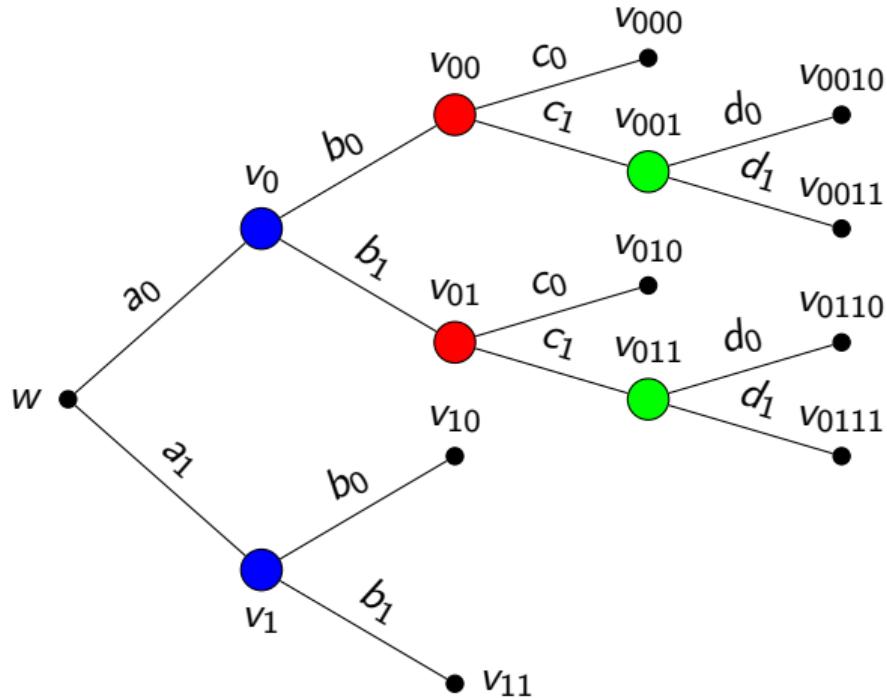
$$d_0 + d_1 = 1$$

Unfolding of events in a cell nature.

Collazo, Görgen, Smith, 2018



Staged trees

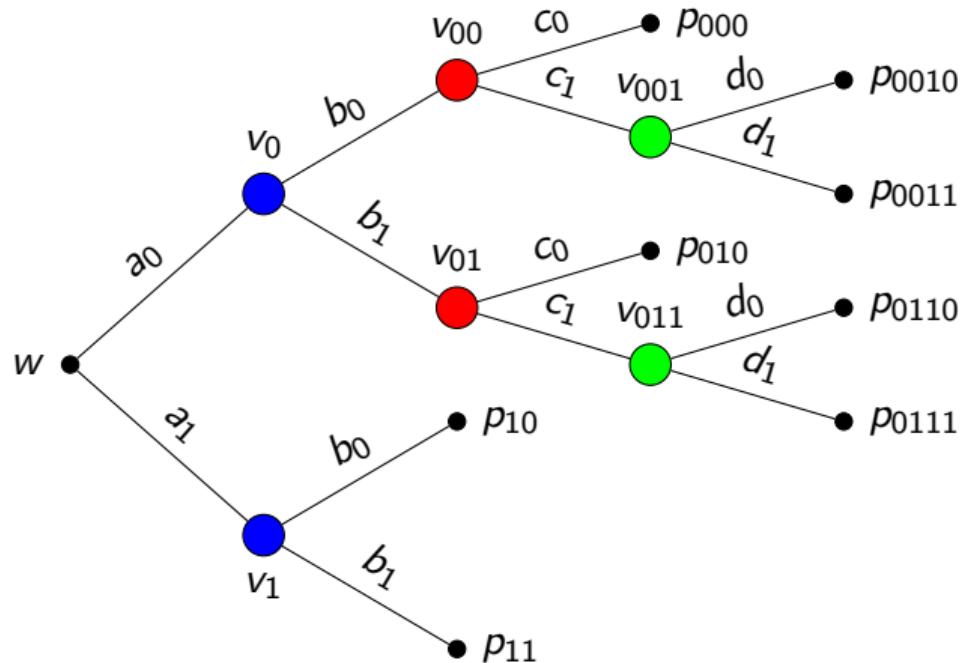


$$\Lambda(\mathcal{T}) = \{\lambda_1, \lambda_2, \dots, \lambda_8\}$$

$$\begin{aligned}\lambda_3 &= (a_0, b_0, c_1, d_1), \\ \lambda_7 &= (a_1, b_0)\end{aligned}$$



Staged trees



$$\Lambda(\mathcal{T}) = \{\lambda_1, \lambda_2, \dots, \lambda_8\}$$

$$p_{000} = a_0 b_0 c_0$$

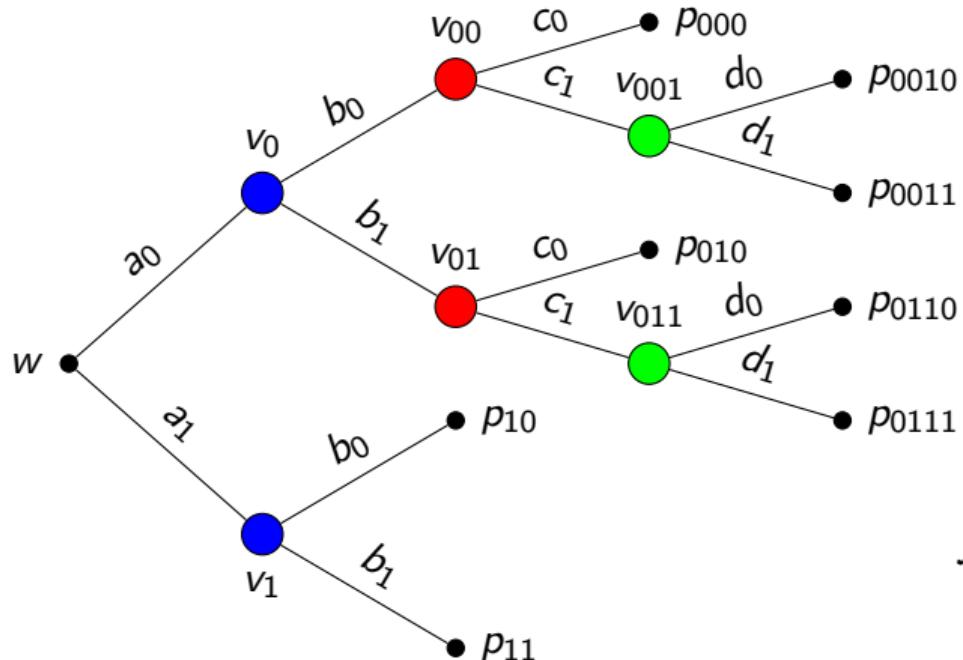
$$p_{0010} = a_0 b_0 c_1 d_0$$

⋮

$$p_{11} = a_1 b_0$$



Staged trees



$$\Lambda(\mathcal{T}) = \{\lambda_1, \lambda_2, \dots, \lambda_8\}$$

$$p_{000} = a_0 b_0 c_0$$

$$p_{0010} = a_0 b_0 c_1 d_0$$

⋮

$$p_{11} = a_1 b_0$$

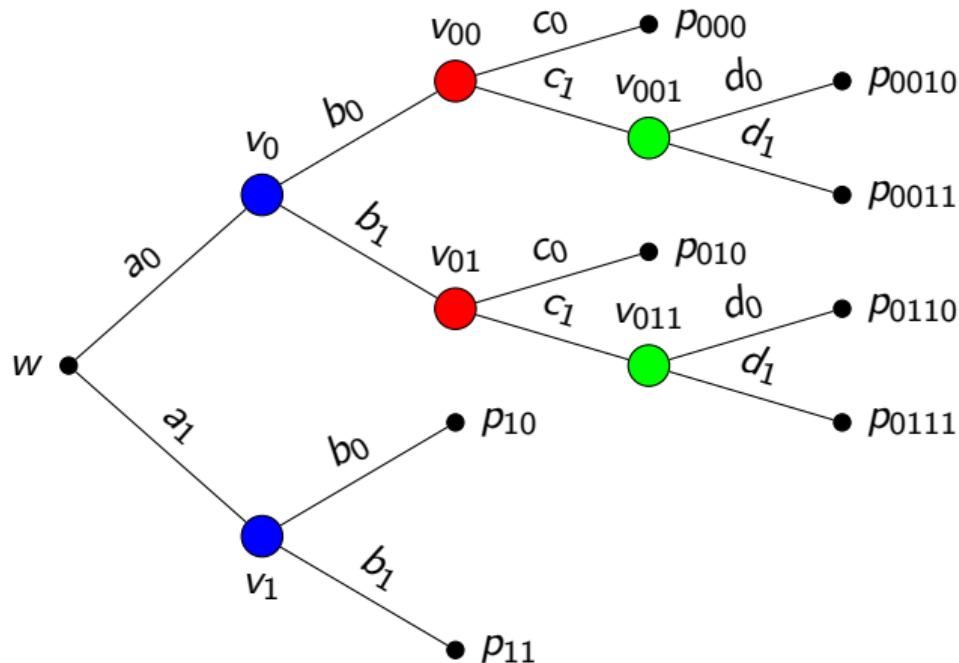
$$p_{000} + p_{0010} + \dots + p_{11} = 1$$

$$\mathcal{M}_{\mathcal{T}} = \{p_{\theta} = (p_{000}, p_{0010}, \dots, p_{11})\}$$

staged tree model



How can we characterize staged trees implicitly?



$$\mathbb{R}[p] = \mathbb{R}[p_{000}, p_{0010}, \dots, p_{11}],$$
$$\mathbb{R}[\Theta] = \mathbb{R}[a_0, a_1, b_0, b_1, \dots]$$

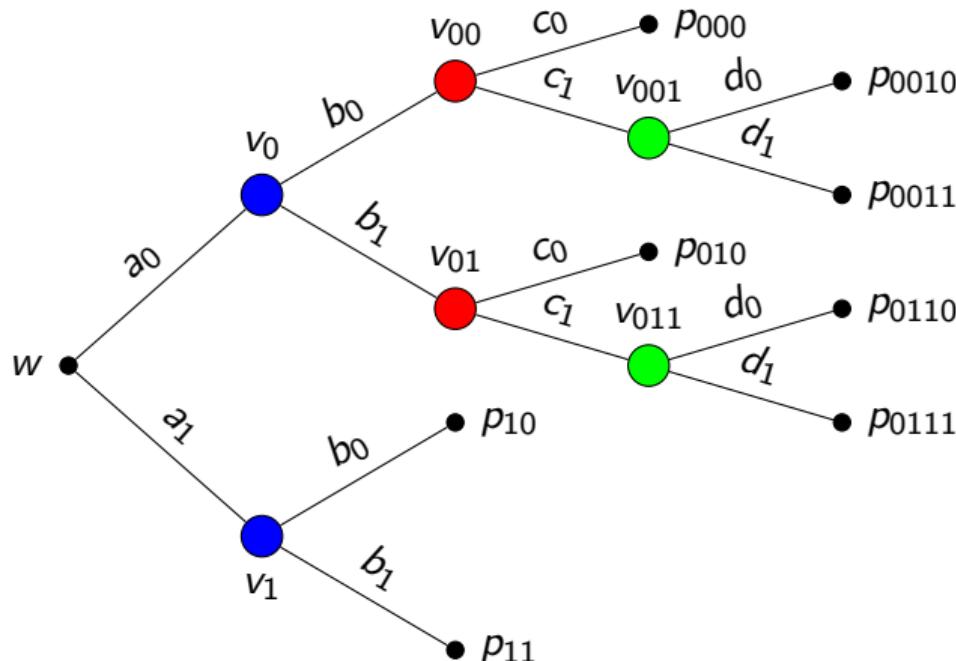
$$\varphi_T : \mathbb{R}[p] \rightarrow \mathbb{R}[\Theta]/\langle \theta - 1 \rangle$$
$$p_i \mapsto \prod_{e \in E(\lambda_i)} \theta(e)$$

$$I_T = \ker(\varphi_T).$$

When is I_T a toric ideal? What are the implicit equations of \mathcal{M}_T ?



Toric Staged Tree

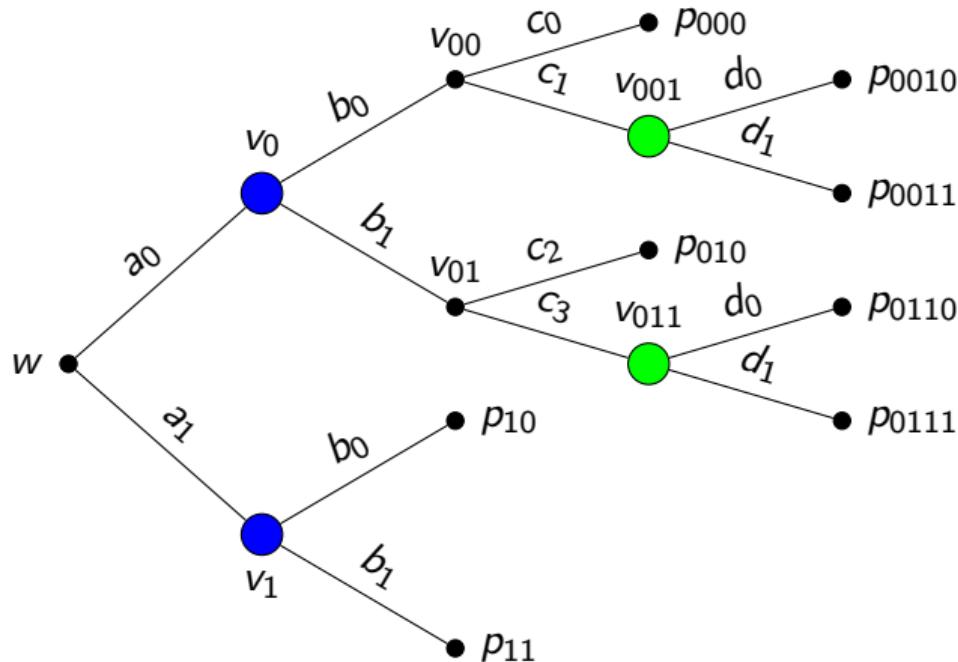


$$\begin{aligned} I_T = \langle & p_{000}p_{11} - p_{010}p_{10}, \\ & p_{0010}p_{11} - p_{0110}p_{10}, \\ & p_{0011}p_{11} - p_{0111}p_{10}, \\ & p_{000}p_{0110} - p_{0010}p_{010}, \\ & p_{000}p_{0111} - p_{0011}p_{010}, \\ & p_{0010}p_{0111} - p_{0011}p_{0110} \rangle. \end{aligned}$$

I_T is prime and
generators are binomials!



Non-Toric Staged Tree



$$\begin{aligned} I_{\mathcal{T}} = & \langle (p_{000} + p_{0010} + p_{0011})p_{11} \\ & - p_{10}(p_{010} + p_{0110} + p_{0111}), \\ & p_{0010}p_{0111} - p_{0011}p_{0110} \rangle. \end{aligned}$$

$I_{\mathcal{T}}$ is prime, but,
not all generators are binomials!

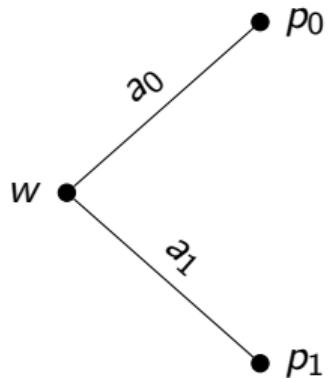


- Necessary conditions for a staged tree to be toric are known
(Duarte & Görgen 2018).
- We implicitly characterize $\mathcal{M}_{\mathcal{T}}$ by doing a Toric Fiber Product (TFP) with tree gluings.
- TFP is a procedure of understanding a complicated ideal from simpler ideals
(Sullivant 2006).
- TFP \Rightarrow Gröbner Bases.
- \mathcal{T} : always toric staged tree.



Gluing Construction of toric staged trees

$\mathcal{T}_1 :$

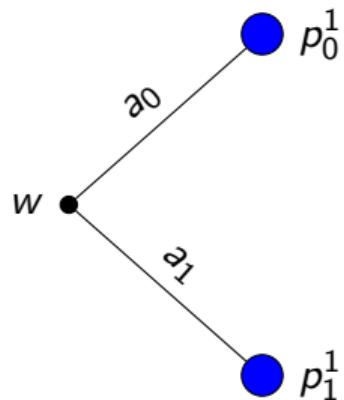


$$\mathbb{R}[p]_{\mathcal{T}_1} = \mathbb{R}[p_0, p_1]$$



Gluing Construction of toric staged trees

$\mathcal{T}_1 :$



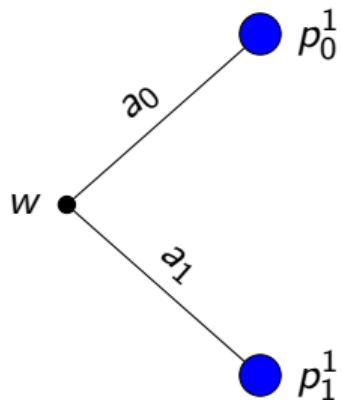
$G = \{G_1 = \{0, 1\}\}$ gluing information

$$\mathbb{R}[p]_{\mathcal{T}_1} = \mathbb{R}[p_0^1, p_1^1]$$



Gluing Construction of toric staged trees

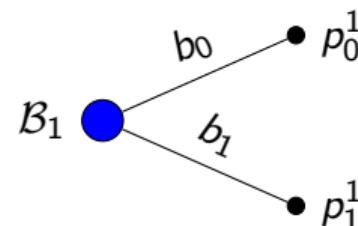
$\mathcal{T}_1 :$



$$\mathbb{R}[p]_{\mathcal{T}_1} = \mathbb{R}[p_0^1, p_1^1]$$

$$G = \{G_1\{0,1\}\}$$

$\mathcal{T}_G :$

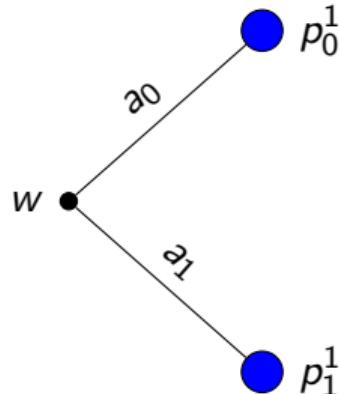


$$\mathbb{R}[p]_{\mathcal{T}_G} = \mathbb{R}[p_0^1, p_1^1]$$



Gluing Construction of toric staged trees

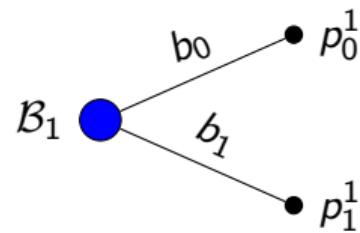
$\mathcal{T}_1 :$



$$\mathbb{R}[p]_{\mathcal{T}_1} = \mathbb{R}[p_0^1, p_1^1]$$

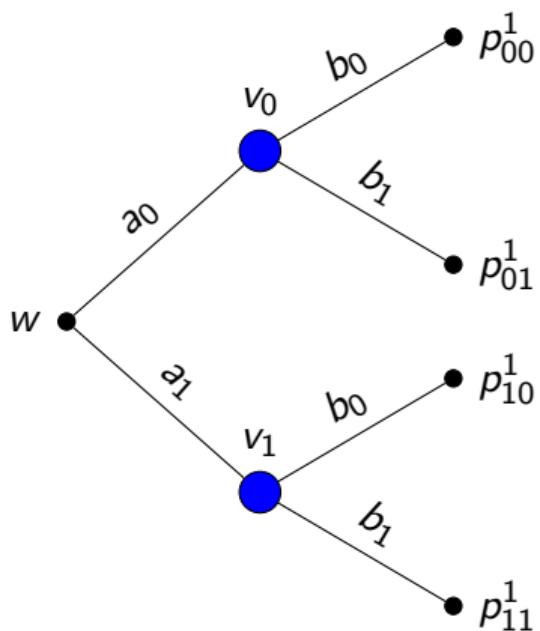
$$G = \{G_1\{0,1\}\}$$

$\mathcal{T}_G :$



$$\mathbb{R}[p]_{\mathcal{T}_G} = \mathbb{R}[p_0^1, p_1^1]$$

$\mathcal{T}_2 :$

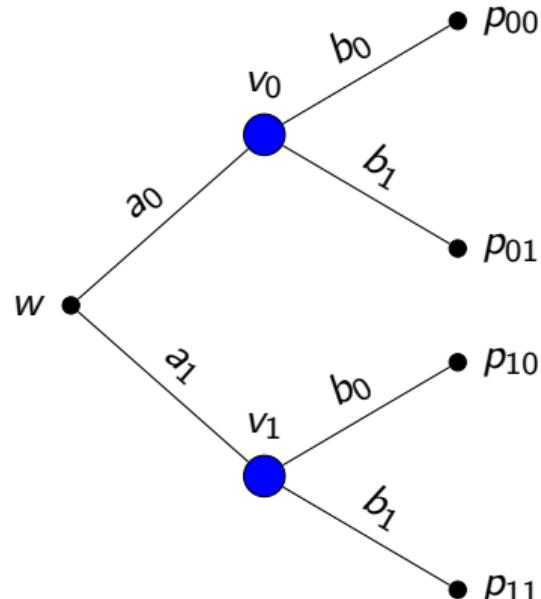


$$\mathbb{R}[p]_{\mathcal{T}_2} = \mathbb{R}[p_{00}^1, p_{01}^1, p_{10}^1, p_{11}^1]$$



Gluing Construction of toric staged trees

$\mathcal{T}_2 :$



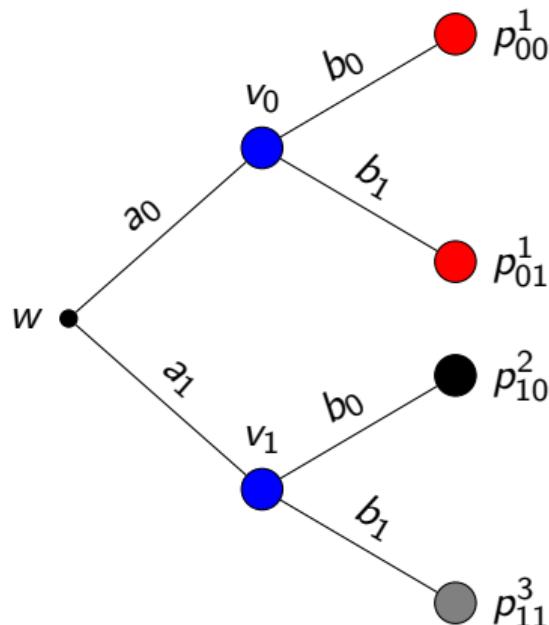
$$I_{\mathcal{T}_2} = I_{\mathcal{T}_1} \times I_{\mathcal{T}_G} = \langle \underline{p_{00}p_{11}} - p_{01}p_{11} \rangle$$

$$\mathbb{R}[p]_{\mathcal{T}_2} = \mathbb{R}[p_{00}, p_{01}, p_{10}, p_{11}]$$



Gluing Construction of toric staged trees

$\mathcal{T}_2 :$



$$G = \{ G_1 = \{00, 01\}, G_2 = \{10\}, G_3 = \{11\} \}$$

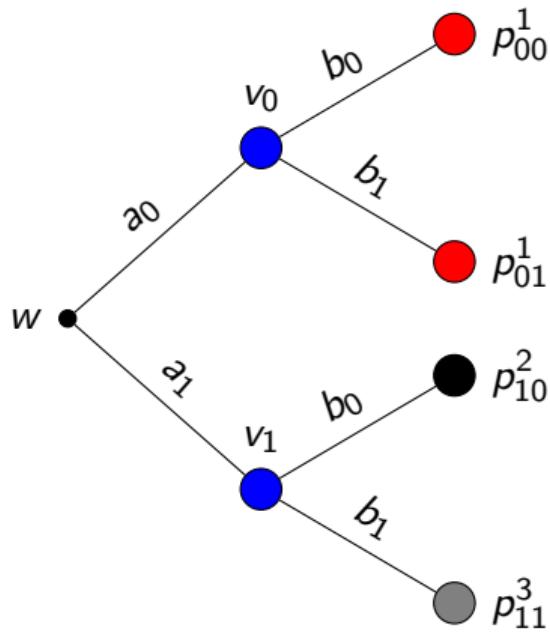
gluing information (**toric compatible**)

$$\mathbb{R}[p]_{\mathcal{T}_2} = \mathbb{R}[p_{00}^1, p_{01}^1, p_{10}^2, p_{11}^3]$$



Gluing Construction of toric staged trees

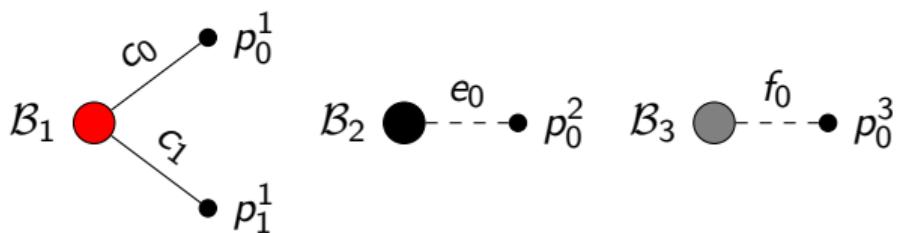
$\mathcal{T}_2 :$



$$\mathbb{R}[p]_{\mathcal{T}_2} = \mathbb{R}[p_{00}^1, p_{01}^1, p_{10}^2, p_{11}^3]$$

$$G = \{ G_1 = \{00, 01\}, G_2 = \{10\}, G_3 = \{11\} \}$$

$\mathcal{T}_G :$

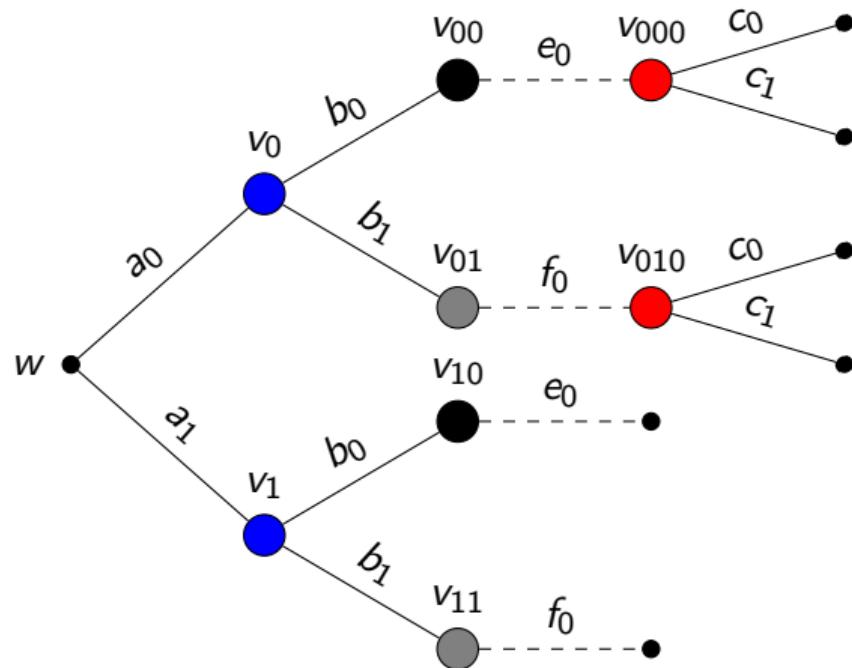


$$\mathbb{R}[p]_{\mathcal{T}_G} = \mathbb{R}[p_0^1, p_1^1, p_0^2, p_0^3]$$



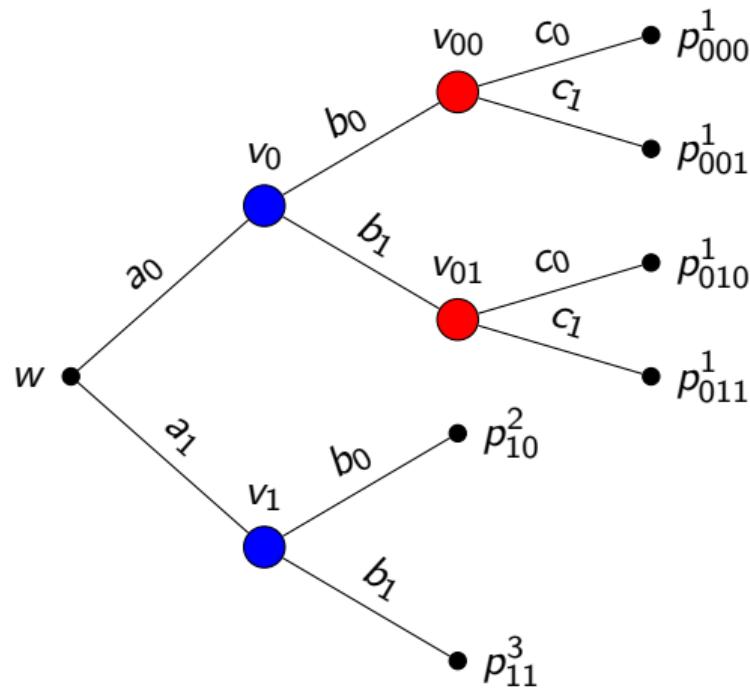
Gluing Construction of toric staged trees

$\mathcal{T}_3 :$



Gluing Construction of toric staged trees

\mathcal{T}_3 :



$$\mathbb{R}[p]_{\mathcal{T}_3} = \mathbb{R}[p_{000}^1, p_{001}^1, p_{010}^1, p_{011}^1, p_{10}^2, p_{11}^3]$$

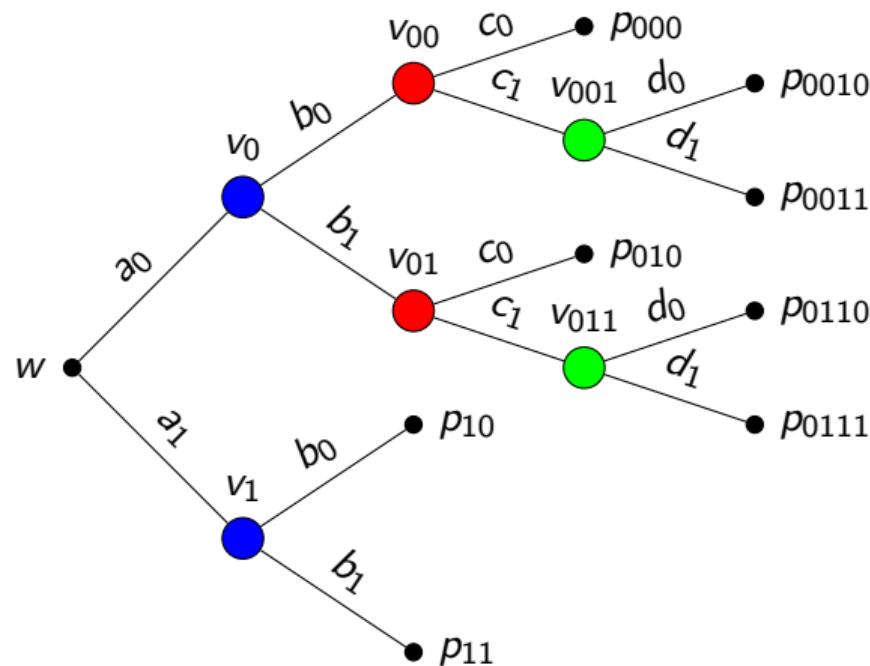
$$\begin{aligned} I_{\mathcal{T}_3} = I_{\mathcal{T}_2} \times I_{\mathcal{T}_G} = & \langle \underline{p_{000}p_{011}} - \underline{p_{010}p_{001}}, \\ & \underline{p_{000}p_{11}} - \underline{p_{010}p_{10}}, \\ & \underline{p_{001}p_{11}} - \underline{p_{011}p_{10}} \rangle \end{aligned}$$

etc...



Gluing Construction of toric staged trees

\mathcal{T} :

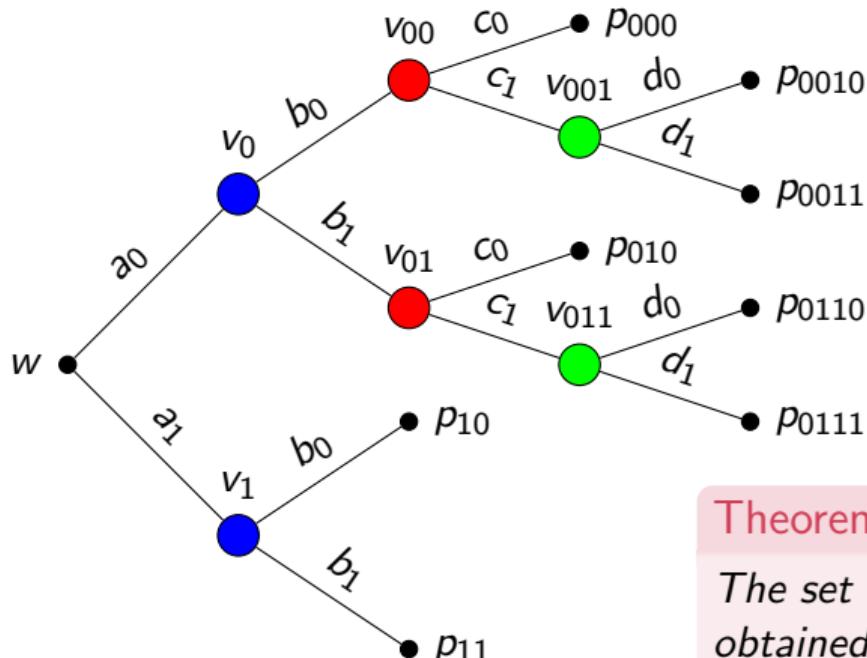


$$I_{\mathcal{T}} = \langle \begin{aligned} & p_{000}p_{0110} - p_{010}p_{0010}, \\ & p_{000}p_{0111} - p_{010}p_{0011}, \\ & p_{000}p_{11} - p_{010}p_{10}, \\ & p_{0010}p_{11} - p_{0110}p_{10}, \\ & p_{0011}p_{11} - p_{0111}p_{10}, \\ & \underline{p_{0010}p_{0111} - p_{0011}p_{0110}} \end{aligned} \rangle$$



Gluing Construction of toric staged trees

\mathcal{T} :



Theorem (Duarte, A. 2019+)

The set of implicit equations of a toric staged tree \mathcal{T} obtained by this gluing construction is a quadratic Gröbner basis.

References

- Görgen C. (2017), *An Algebraic Characterisation of Staged Trees*, PhD thesis, University of Warwick, Department of Statistics,
- Collazo R.A., Görgen C., Smith J.Q. (2018), *Chain Event Graphs*, Chapman & Hall.
- Duarte E., Görgen C. (2018), *Equations Defining Probability Tree Models*, arXiv: 1820.04511,
- Sullivant S. (2007), *Toric Fiber Products*, J. Algebra **316**, no. 2, 560-577.



References

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Thank you for your attention!

