

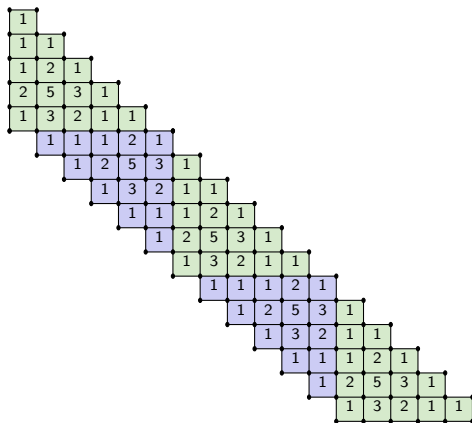
# Frieze patterns and the Grassmannian

Jordan McMahon

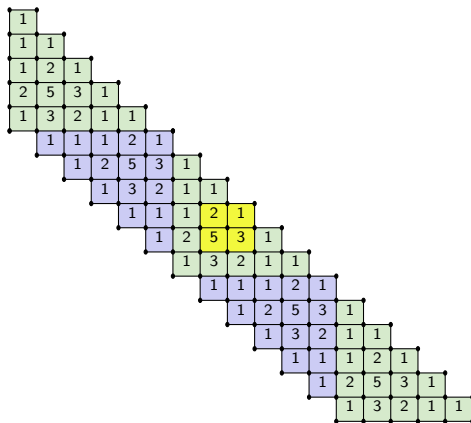
University of Graz

February 19, 2019

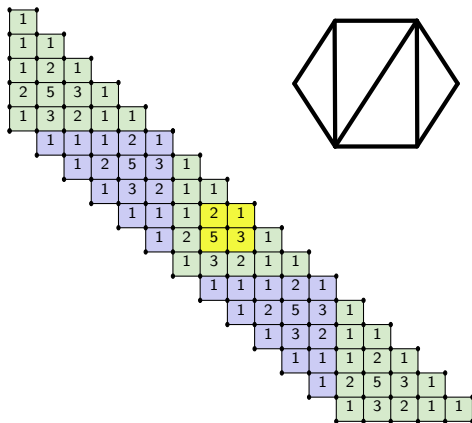
# Conway-Coxeter Frieze patterns



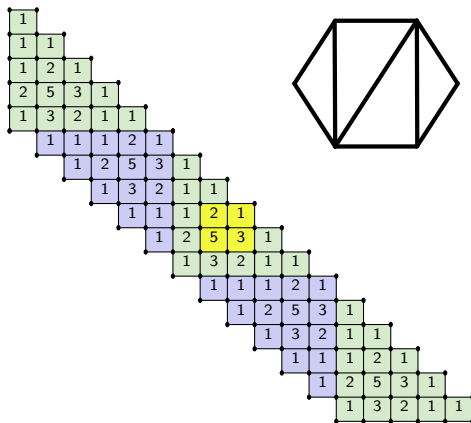
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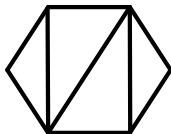
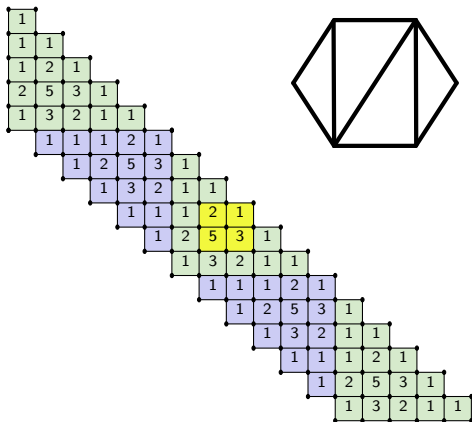
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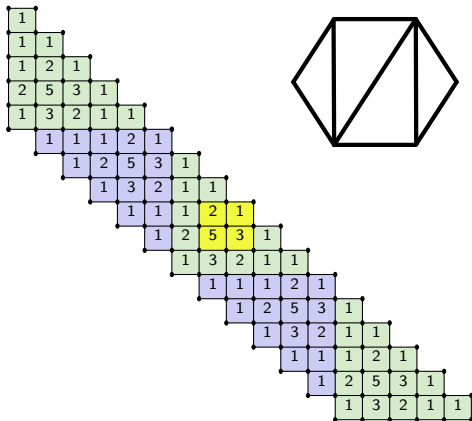
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- all top and bottom entries are 1.

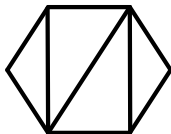
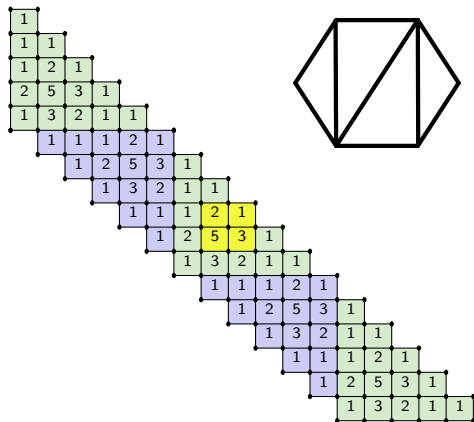
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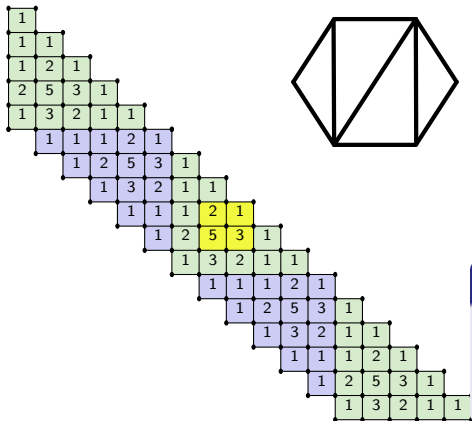
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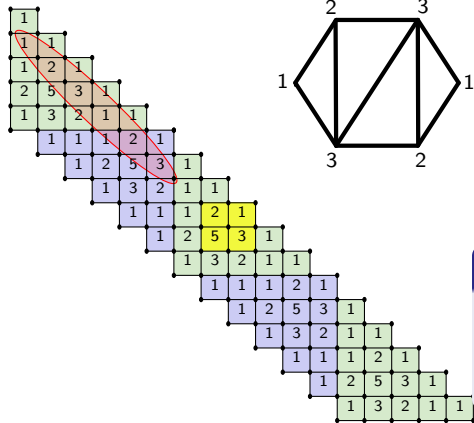
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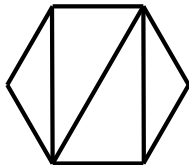


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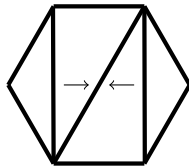
$$0 \longrightarrow k \xleftarrow{1} k \xrightarrow{1} k \longrightarrow 0 \longrightarrow 0$$

# Representation theory and triangulations

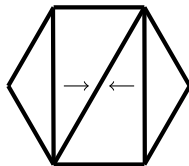
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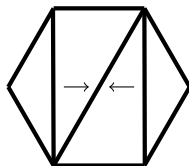


## Theorem

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- $\{ \text{Triangulations of convex polygons with no internal triangles} \}$
- $\{ \text{Quivers of type A} \}$

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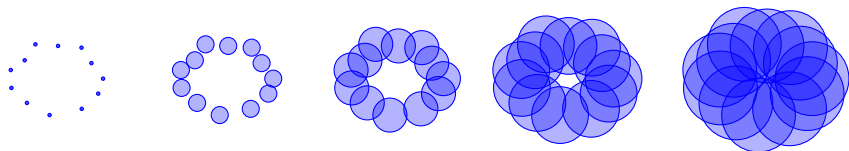
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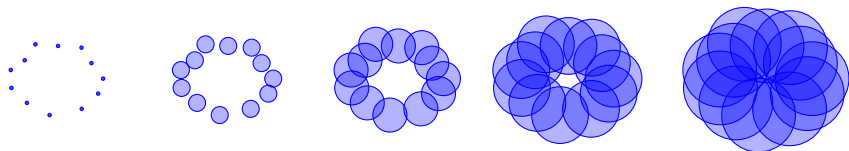
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In fact, each frieze pattern displays the indecomposable representations of a given type  $A$  quiver, and the number of their subrepresentations!

# Persistence homology

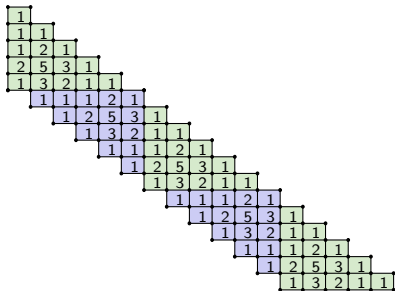


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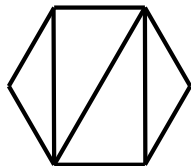


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## Conway-Coxeter frieze patterns



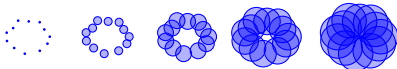
## Triangulations of a polygon



## Representation of type A quiver

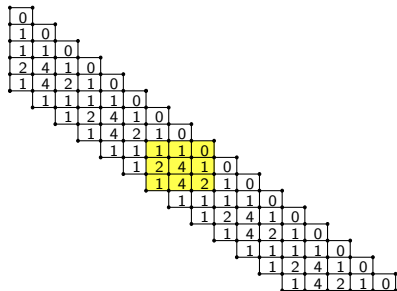
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## Zig-zag persistence homology



# Multi-dimensional versions

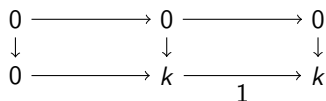
## $SL_k$ Frieze Patterns



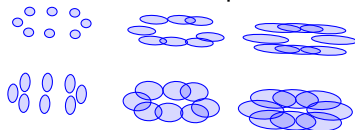
## cluster in Grassmannian

$\{124, 134, 145, 146\}$

## Representation of type $A \otimes A$ quiver



## Commutative ladder persistence



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- For example if  $k = 3, n = 6$  there is a cluster

$$\{124, 134, 145, 146, \\ 123, 234, 345, 456, 561, 612\}$$

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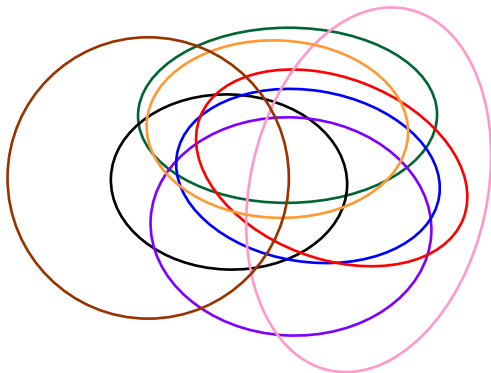
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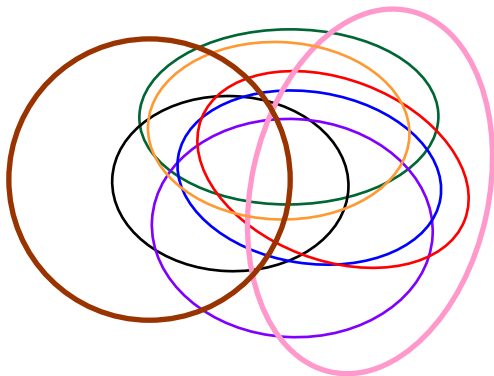
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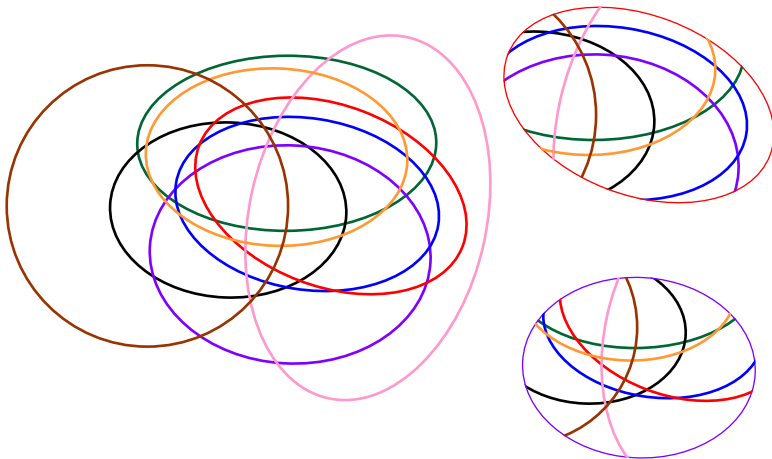
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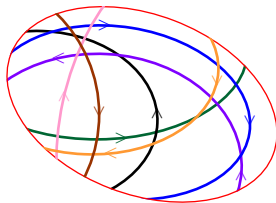
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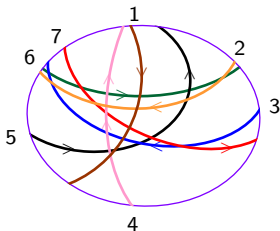
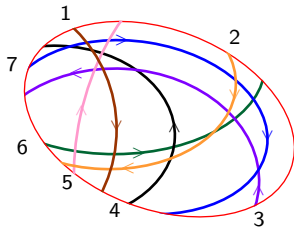
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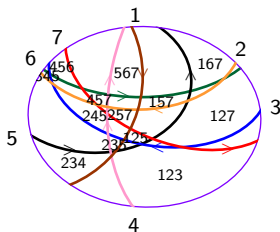
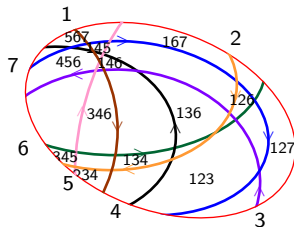
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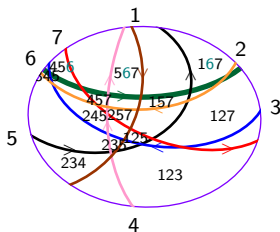
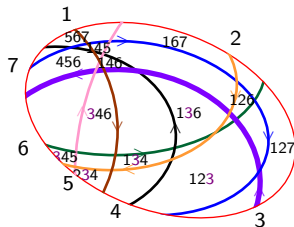
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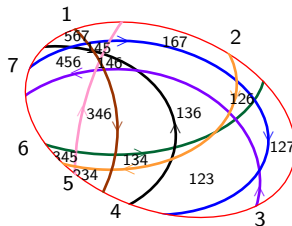


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$\{134, 346, 136, 146, 145, 126\}$



$\{125, 235, 157, 257, 245, 457\}$

