

Convex Cones Spanned by Regular Polytopes, and Probabilistic Applications

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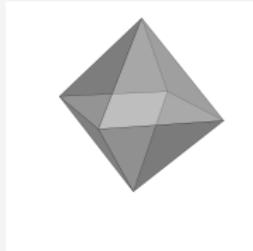
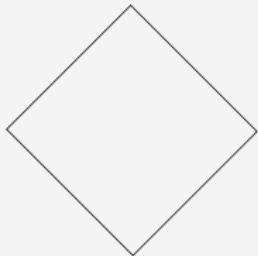
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We are interested in three types of regular polytopes

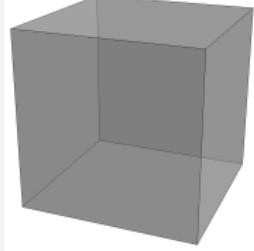
Cross-polytope

$$\text{Conv}\{\pm e_1, \dots, \pm e_n\}$$



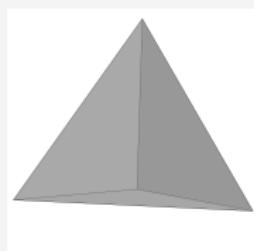
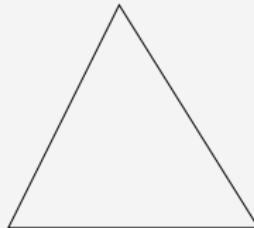
Cube

$$[-1, 1]^n$$



Simplex

$$\text{Conv}\{e_1, \dots, e_{n+1}\}$$



Definition: Polyhedral cone

A polyhedral cone $C \subset \mathbb{R}^n$ is the positive hull of finitely many vectors.

$$C = \text{pos}(v_1, \dots, v_k)$$

$$\text{pos}(v_1, \dots, v_k) := \left\{ \sum_{j=1}^k a_j v_i : a_i \geq 0 \right\}$$

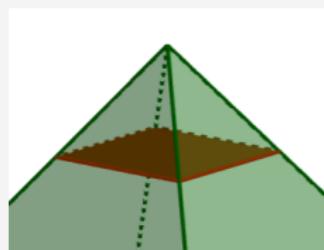
For the rest of the talk when we speak of "cones" it will be short for polyhedral cones

Definition of our cones

Cross-polytope

$$C_n^\Diamond(\sigma^2) :=$$

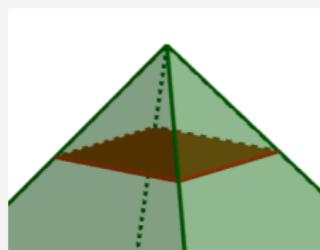
$$\text{pos}(\sigma e_{n+1} \pm e_j : j \in \{1, \dots, n\})$$



Cube

$$C_n^{\boxplus}(\sigma^2) :=$$

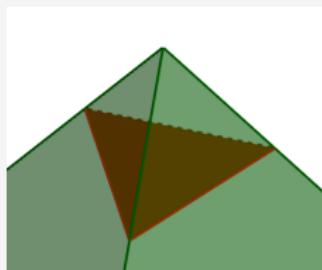
$$\text{pos}(\sigma e_{n+1} + \sum_{j=1}^n \varepsilon_j e_j : \varepsilon \in \{\pm 1\}^n)$$



Simplex

$$C_n^\Diamond(\sigma^2) :=$$

$$\text{pos}(\sigma e_{n+1} + e_j : 1 \leq j \leq n)$$



Polar cones



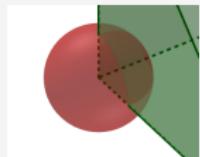
- ▶ $C \subset \mathbb{R}^n$ polyhedral cone
- ▶ Polar cone $C^o = \{v \in \mathbb{R}^n : \langle v, w \rangle \leq 0 \text{ for every } w \in C\}$

Proposition (Kabluchko, S.; 2019+)

$$(C_n^\diamond (\sigma^2))^o \cong C_n^{\boxdot} \left(\frac{1}{\sigma^2} \right)$$

$$(C_n^{\boxdot} (\sigma^2))^o \cong C_n^\diamond \left(\frac{1}{\sigma^2} \right)$$

Angles of n -dimensional cones



- ▶ $B_r(0)$ any n -dimensional ball centred in 0.
- ▶ Inner solid angle $\alpha(C)$ of n -dimensional cone $C \subset \mathbb{R}^n$: amount of $B_r(0)$ that is in the cone.

$$\alpha(C) = \frac{\lambda(C \cap B_r(0))}{\lambda(B_r(0))}$$

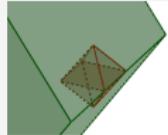
$$\alpha(C) = \mathbb{P}(\xi \in C), \xi \sim \mathcal{N}^n(0, 1)$$

Solid angles of these cones

$\xi = (\xi_1, \dots, \xi_{n+1}) \sim \mathcal{N}^{n+1}(0, 1) \Rightarrow \xi_j \sim \mathcal{N}^1(0, 1)$ independent

$$\begin{aligned}
\alpha(C_n^\Phi(\sigma^2)) &= \mathbb{P}(\xi \in C_n^\Phi(\sigma^2)) = \mathbb{P}\left(\forall v \in C_n^\boxplus\left(\frac{1}{\sigma^2}\right) : \langle \xi, v \rangle \leq 0\right) \\
&= \mathbb{P}\left(\left\langle \xi, \frac{1}{\sigma}e_{n+1} + \sum_{j=1}^n \varepsilon_j e_j \right\rangle \leq 0 \text{ for every } \varepsilon \in \{-1, 1\}^n\right) \\
&= \mathbb{P}\left(\frac{1}{\sigma}\xi_{n+1} + \sum_{j=1}^n \varepsilon_j \xi_j \leq 0\right) \\
&= \mathbb{P}\left(\frac{1}{\sigma}\xi_{n+1} \leq 0, \left|\frac{1}{\sigma}\xi_{n+1}\right| \geq \sum_{j=1}^n |\xi_j|\right) = \mathbb{P}\left(\frac{1}{\sigma}\xi_{n+1} \geq \sum_{j=1}^n |\xi_j|\right) \\
\alpha(C_n^\boxplus(\sigma^2)) &= \mathbb{P}\left(\frac{1}{\sigma}\xi_{n+1} \geq \max_{1 \leq j \leq n} |\xi_j|\right)
\end{aligned}$$

Tangent cones and normal cones



- ▶ F face of polytope $P \subset \mathbb{R}^n$ or cone $C \subset \mathbb{R}^n$
- ▶ Tangent cone $T_F(P) = \{v \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ such that } f + \varepsilon v \in P\}$ for $f \in \text{relint}(F)$
- ▶ Normal cone $N_F(P) = (T_F(P))^\circ$

Proposition (Kabluchko, S.; 2019+)

$P \subset \mathbb{R}^n$ cross-polytope of dimension n , F face of codimension l

$$\alpha(T_F(P)) = \alpha \left(C_{l-1}^\oplus \left(\frac{1}{n-l+1} \right) \right) = \mathbb{P} \left(\sqrt{n-l+1} \xi_l \geq \sum_{j=1}^{l-1} |\xi_j| \right)$$

$$\alpha(N_F(P)) = \alpha \left(C_{l-1}^\boxplus (n-l+1) \right) = \mathbb{P} \left(\xi_l \geq \sqrt{n-l+1} \max_{1 \leq j \leq l-1} |\xi_j| \right)$$

$(\xi_i)_{i \in \mathbb{N}} \sim \mathcal{N}(0, 1)$ independent

Definition: k^{th} intrinsic volume

- ▶ C polyhedral cone
- ▶ $0 \leq k \leq \dim(C)$

$$\nu_k(C) = \sum_{F \text{ is } k\text{-face of } C} \alpha(F)\alpha(N_F(C))$$

- ▶ $\nu_k(C) = 0$ if $k > \dim(C)$

k^{th} intrinsic volumes of our cones

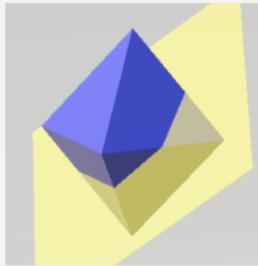
Theorem (Kabluchko, S.; 2019+)

$$0 \leq k \leq n$$

$$\nu_k(C_n^\Phi(\sigma^2)) = 2^k \binom{n}{k} \alpha\left(C_{n-k}^{\boxplus} \left(\frac{1}{\sigma^2} + k\right)\right) \alpha\left(C_k^\Phi(\sigma^2)\right)$$

$$\nu_k\left(C_n^{\boxplus}(\sigma^2)\right) = 2^{n-k+1} \binom{n}{k-1} \alpha\left(C_{k-1}^{\boxplus}(\sigma^2 + n - k + 1)\right) \alpha\left(C_{n-k+1}^\Phi\left(\frac{1}{\sigma^2}\right)\right)$$

Application: Random sections of regular polytopes



- ▶ P regular polytope centred in 0
- ▶ $S \subset \mathbb{R}^n$ random k -dimensional linear subspace chosen uniformly from the Grassmannian $Gr(k, \mathbb{R}^n)$, i.e. the set of all subspaces with dimension k .
- ▶ $\phi^P(j, k, n)$ number of j -faces of $P \cap S$, $0 \leq j < k < n$
- ▶ We are interested in $\mathbb{E}\phi^P(j, k, n)$.
- ▶ Introduced by [Lonke, 2000]

P regular polytope centred in 0

Lemma (Kabluchko, S.; 2019+)

$$\phi^P(j, n-l, n) = \sum_{F \text{ is } (j+l)\text{-face of } P} 1_{\{F \cap S \neq \emptyset\}} \text{ almost surely,}$$

i.e. number of j -faces of $P \cap S$ a.s. equals number of $(j+l)$ -faces of P that are intersected by S .

$$\mathbb{E}\phi^P(j, n-l, n) = |\{F : F \text{ is } (j+l)\text{-face of } P\}| \cdot \mathbb{P}(F \cap S \neq \emptyset)$$

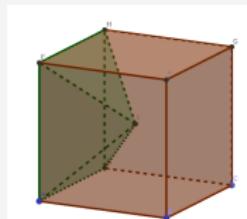
$$\mathbb{E}\phi^P(j, k, n) = |\{F : F \text{ is } (j+n-k)\text{-face of } P\}| \cdot \mathbb{P}(F \cap S \neq \emptyset)$$

- ▶ P regular polytope centred in 0
- ▶ C convex cone spanned by face F

$$F \cap S \neq \emptyset \Leftrightarrow C \cap S \neq \{0\}$$

Conic Crofton formula:

$$\mathbb{P}(C \cap S \neq \{0\}) = 2(\nu_{l+1}(C) + \nu_{l+3}(C) + \dots), l = \text{codimension of } S$$



Results

- ▶ $\mathbb{E}\phi^P(j, k, n) = |\{F : F \text{ is } (j+n-k)\text{-face of } P\}| \cdot \mathbb{P}(C \cap S \neq \{0\})$
- ▶ $\mathbb{P}(C \cap S \neq \{0\}) = 2(\nu_{l+1}(C) + \nu_{l+3}(C) + \dots)$, l codimension of S

Cube:

$$C \cong C_{j+n-k}^{\boxtimes}(k-j) \quad C \cong C_{j+n-k+1}^{\Diamond}(-\frac{1}{n+1}) \quad C \cong C_{j+n-k+1}^{\Diamond}(0)$$

Simplex:

Cross-polytope:

Generalisation of C_n^{\Diamond} to Parameters smaller 0: Define the cone by the scalar products of the vectors spanning it

- ▶ $\mathbb{E}\phi^{\boxtimes}(j, k, n) = 2^{k-j+1} \binom{n}{n-k+j} (\nu_{n-k+1}(C_{n-k+j}^{\boxtimes}(k-j)) + \nu_{n-k+3}(C_{n-k+j}^{\boxtimes}(k-j)) + \dots)$
- ▶ $\mathbb{E}\phi^{\Diamond}(j, n-l, n) = 2 \binom{n+1}{j+l+1} \cdot (\nu_{l+1}(C_{j+l+1}^{\Diamond}(-\frac{1}{n+1})) + \nu_{l+3}(C_{j+l+1}^{\Diamond}(-\frac{1}{n+1})) + \dots)$
- ▶ $\mathbb{E}\phi^{\Diamond}(j, n-l, n) = 2 \binom{n}{j+l+1} \left(\binom{j+l}{l} + \binom{j+l}{l+1} + \dots \right)$

Random Sections and Random Projections

- ▶ $P \subset \mathbb{R}^n$ Polytope, $S \subset \mathbb{R}^n$ random linear subspace of dimension k as before.
- ▶ $\Pi_S P$ orthogonal projection of P onto S .
- ▶ $\#\mathcal{F}_{k-j-1}(S \cap P^\circ) = \#\mathcal{F}_j(\Pi_S P)$
- ▶ $\mathbb{E}\phi^\triangleleft(j, k, n)$ found by Affentranger, Schneider ('92)
- ▶ $\mathbb{E}\phi^{\boxtimes}(j, k, n), \mathbb{E}\phi^\diamondsuit(j, k, n)$ found by Böröczky, Henk ('99)

Gaussian Polytopes

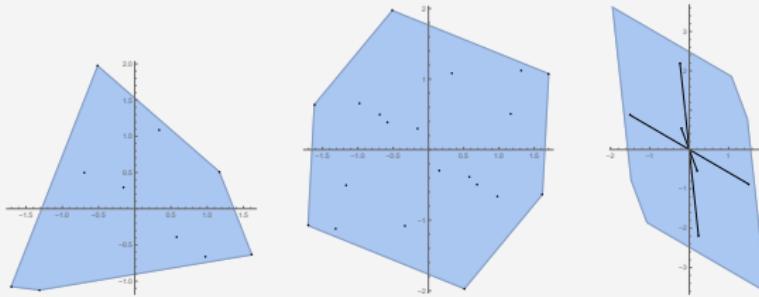
- ▶ $X_1, X_2, \dots \sim \mathcal{N}^d(0, 1)$ i.i.d
- ▶ $P_n^{\text{Gauss}} := \text{Conv}\{X_1, \dots, X_n\}$, $d < n$ Gaussian polytope
- ▶ Similar polytopes:
 - ▶ Symmetric Gaussian polytope:

$$P_n^{\text{symm}} := \text{Conv}\{\pm X_1, \dots, \pm X_n\}, \quad d < n$$

- ▶ Gaussian zonotope:

$$P_n^{\text{zonotope}} := [-X_1, X_1] + \dots + [-X_n, X_n], \quad d < n$$

(For $x, y \in \mathbb{R}^d$ define $[x, y] := \text{Conv}\{x, y\} \subset \mathbb{R}^d$)



Absorption Probability

- ▶ $P_n^{\text{Gauss}} := \text{Conv}\{X_1, \dots, X_n\} \subset \mathbb{R}^d$, $d < n$ Gaussian polytope
- ▶ We are interested in the absorption probability $\mathbb{P}[x \in P_n^{\text{Gauss}}], x \in \mathbb{R}^d$.
- ▶ $\mathcal{N}^d(0, 1)$ invariant under rotations
⇒ Better definition: $f_n^{\text{Gauss}}(|x|) := \mathbb{P}[x \notin P_n^{\text{Gauss}}]$, i.e. for $\sigma \geq 0$:

$$f_n^{\text{Gauss}}(\sigma) := \mathbb{P}[\sigma e_1 \notin P_n^{\text{Gauss}}]$$

- ▶ Non-absorption probability $f_n^{\text{Gauss}} : [0, \infty) \rightarrow [0, 1]$

Absorption Probability - f and p

- ▶ $P_n^{\text{Gauss}} := \text{Conv}\{X_1, \dots, X_n\}$, $d < n$ Gaussian polytope
- ▶ $X \sim \mathcal{N}^d(0, 1)$ independent of X_1, \dots, X_n
- ▶ $p_n^{\text{Gauss}}(\sigma) := \mathbb{P}[\sigma X \notin P_n^{\text{Gauss}}]$ easier to compute.

Absorption Probability - Calculations

$$\begin{aligned}
 p_n^{\text{symm}}(\sigma) &= \mathbb{P}[\sigma X \notin P_n^{\text{symm}}] = \mathbb{P}[-\sigma X \notin P_n^{\text{symm}}] = \mathbb{P}[0 \notin \text{Conv}\{\pm X_1 + \sigma X, \dots, \pm X_n + \sigma X\}] \\
 &= \mathbb{P}\left[\left(0 = \sum_{i=1}^n \lambda_i X_i + \lambda \sigma X \text{ for } \lambda_1, \dots, \lambda_n \in \mathbb{R}, \lambda \geq \sigma \sum_{i=1}^n |\lambda_i|\right) \Rightarrow \lambda = \lambda_1 = \dots = \lambda_n = 0\right] \\
 &= \mathbb{P}[C \cap U = \{0\}]
 \end{aligned}$$

with

$$\begin{aligned}
 U &:= \left\{ (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n y_i X_i + y_{n+1} X = 0 \right\} \\
 C &:= \left\{ (\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}^{n+1} : \lambda_{n+1} \geq \sigma \sum_{i=1}^n |\lambda_i| \right\}.
 \end{aligned}$$

- ▶ Short calculation $\Rightarrow C \cong C_n^\Phi(\sigma^2)$
- ▶ U random hyperplane uniformly distributes on $\text{Gr}(n, \mathbb{R}^{n+1})$.
- ▶ Conic Crofton formula can be applied.

Absorption Probability - Results for p

The two other cases are very similar.

$$p_{n,d}^{\text{Gauss}}(\sigma^2) = \mathbb{P}[\sigma X \notin P_n^{\text{Gauss}}] = 2(\nu_{d-1}(C_n^{\triangle}(\sigma^2)) + \nu_{d-3}(C_n^{\triangle}(\sigma^2)) + \dots)$$

$$p_{n,d}^{\text{symm}}(\sigma^2) = \mathbb{P}[\sigma X \notin P_n^{\text{symm}}] = 2(\nu_{d-1}(C_n^{\diamondsuit}(\sigma^2)) + \nu_{d-3}(C_n^{\diamondsuit}(\sigma^2)) + \dots)$$

$$p_{n,d}^{\text{zonotope}}(\sigma^2) = \mathbb{P}[\sigma X \notin P_n^{\text{zonotope}}] = 2(\nu_{d-1}(C_n^{\boxplus}(\sigma^2)) + \nu_{d-3}(C_n^{\boxplus}(\sigma^2)) + \dots)$$

Absorption Probability - Connection between f and p

Proposition (Kabluchko, Zaporozhets; 2017; Kabluchko, S.; 2019+)

- ▶ Gaussian polytope, symmetric Gaussian polytope or Gaussian zonotope
- ▶ Non-absorption probability $f_n^P: [0, \infty) \rightarrow [0, 1]$, $P \in \{\text{Gauss, symm, zonotope}\}$ satisfies

$$\int_0^\infty f_n^P(\sqrt{2u}) u^{\frac{d}{2}-1} e^{-\lambda u} du = \Gamma\left(\frac{d}{2}\right) \lambda^{-\frac{d}{2}} p_n^P\left(\frac{1}{\lambda}\right)$$

for every $\lambda > 0$.

For $d = 2$ we have

$$\int_0^\infty f_n^P(\sqrt{2u}) e^{-\lambda u} du = \frac{1}{\lambda} p_n^P\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda} \nu_1\left(C_n^P\left(\frac{1}{\lambda}\right)\right).$$

Gaussian polytope:

$$\begin{aligned} \int_0^\infty f_{n,2}^{\text{Gauss}}(\sqrt{2u}) e^{-\lambda u} du &= \dots = \frac{\sqrt{\lambda+1}}{\lambda} \int_0^\infty e^{-\lambda t} \frac{d}{dt} (\Phi^n(\sqrt{2t}) - \Phi^n(-\sqrt{2t})) dt \\ \Rightarrow f_{n,2}^{\text{Gauss}}(\sqrt{2u}) &= \mathbb{P}[M_n^2 + \xi^2 \leq 2u] + \frac{d}{du} \mathbb{P}[M_n^2 + \xi^2 \leq 2u] \\ \xi, \xi_1, \dots, \xi_n &\sim \mathcal{N}(0, 1) \text{ i.i.d.}, \quad M_n := \max\{\xi_1, \dots, \xi_n\} \end{aligned}$$

Symmetric Gaussian polytope:

$$\begin{aligned} \int_0^\infty f_{n,2}^{\text{symm}}(\sqrt{2u}) e^{-\lambda u} du &= \dots = \frac{\sqrt{\lambda+1}}{\lambda} \int_0^\infty e^{-\lambda t} \left(\frac{d}{dt} (2\Phi(\sqrt{2t}) - 1)^n \right) dt \\ \Rightarrow f_{n,2}^{\text{symm}}(\sqrt{2u}) &= \mathbb{P}[M_n^{\text{sqr}} + \xi^2 \leq 2u] + \frac{d}{du} \mathbb{P}[M_n^{\text{sqr}} + \xi^2 \leq 2u] \\ \xi, \xi_1, \dots, \xi_n &\sim \mathcal{N}(0, 1) \text{ i.i.d.}, \quad M_n^{\text{sqr}} := \max\{\xi_1^2, \dots, \xi_n^2\} \end{aligned}$$

Gaussian zonotope:

$$\int_0^\infty f_{n,2}^{\text{symm}}(\sqrt{2u}) e^{-\lambda u} du = 2^n \sqrt{\frac{1+n\lambda}{\pi\lambda^3}} \int_0^\infty \frac{e^{-nt}}{\sqrt{t}} \Re \left[\left(\frac{1}{2} + \frac{i}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{z^2} dz \right)^n \right] e^{-\frac{t}{\lambda}} dt.$$

References

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Random projections of regular simplices

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Random projections of regular polytopes

Archiv der Mathematik (1999)

KABLUCHKO and ZAPOROZHETS

Absorption probabilities for Gaussian polytopes, and regular spherical simplices

arXiv preprint arXiv:1704.04968 (2017)

LONKE

On random sections of the cube

Discrete & Computational Geometry (2000)