The monic rank
and instances of Shapiro’s Conjecture

Arthur Bik
University of Bern

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j.w.w. Jan Draisma, Alessandro Oneto and Emanuele Ventura

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Examples of rank

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Definition

The rank of a vector $v \in V$ is the minimal $r \in \mathbb{Z}_{\geqslant 0}$ such that

$$v = w_1 + \cdots + w_r, \quad w_i \in X \backslash \{0\}$$

where $X \subseteq V$ is the cone of vectors of rank $\leqslant 1$.

Examples

- $V = \mathbb{C}^{n \times m}$ and $X = \{vw^T \mid v \in \mathbb{C}^n, w \in \mathbb{C}^m\}$
- $V = \{A \in \mathbb{C}^{n \times n} \mid A = A^T\}$ and $X = \{vv^T \mid v \in \mathbb{C}^n\}$
- $V = \{A \in \mathbb{C}^{n \times n} \mid \operatorname{tr} A = 0\}$ and $X = \{vw^T \mid v, w \in \mathbb{C}^n, w^T v = 0\}$
- $V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ and $X = \{v_1 \otimes v_2 \otimes v_3 \mid v_1, v_2, v_3 \in \mathbb{C}^n\}$
- $V = \mathbb{C}[x, y]_{(de)}$ and $X = \{f^d \mid f \in \mathbb{C}[x, y]_{(e)}\}$

Examples of upper bounds on the rank

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- The maximum rank of an $n \times m$ matrix equals $\min(n, m)$.
- The maximum rank of a symmetric $n \times n$ matrix equals n.
- The maximum rank of a trace-zero $n \times n$ matrix equals n.
- The maximum rank of a $2 \times 2 \times 2$ tensor equals 3.

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Conjecture (Boris Shapiro)

Every homogeneous binary form of degree $d \cdot e$ is the sum of at most d d-th powers of forms of degree e.

Known: True for $e = 1$, for $d = 1, 2$ and for $(d, e) = (3, 2)$.

Special case: rank $\leqslant 1$ = powers of linear forms

u^b

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Consider

$$(x + a_1 y)^d + (x + a_2 y)^d + \cdots + (x + a_d y)^d$$

$$=$$

$$dx^d + \binom{d}{1} b_1 x^{d-1} y + \binom{d}{2} b_2 x^{d-2} y^2 + \cdots + \binom{d}{d} b_d y^d$$

with $b_k = a_1^k + \cdots + a_d^k$.

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with $b_k = a_1^k + \cdots + a_d^k$.

Known: The map $(a_1, \ldots, a_d) \mapsto (b_1, \ldots, b_d)$ is a finite morphism.

Special case: rank $\leqslant 1$ = powers of linear forms

Consider

$$(x + a_1 y)^d + (x + a_2 y)^d + \cdots + (x + a_d y)^d$$

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Known: The map $(a_1, \ldots, a_d) \mapsto (b_1, \ldots, b_d)$ is a finite morphism.

Using coordinate transformations, this implies:

$$\mathbb{C}[x, y]_{(d)} = \left\{ \ell_1^d + \cdots + \ell_d^d \;\middle|\; \ell_1, \ldots, \ell_d \in \mathbb{C}[x, y]_{(1)} \right\}$$

The monic rank

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Let V be a finite-dimensional vector space.

Let $X \subseteq V$ be a non-degenerate irreducible Zariski-closed cone.

Let $h\colon V \to \mathbb{C}$ be a non-zero linear function and take $H = h^{-1}(1) \subseteq V$.

Definition

The monic rank of a vector $v \in V \setminus h^{-1}(0)$ is the minimal r such that

$$\frac{r}{h(v)} \cdot v = w_1 + \cdots + w_r$$

with $w_1, \ldots, w_r \in X \cap H$.

Examples of monic rank

We keep V and X the same.

And, we define what is means to be monic:

- An $n \times m$ matrix is monic when its top-left coefficient equals 1.
- symmetric $n \times n$ matrices $\rightsquigarrow$ top-left coefficient
- skew-symmetric $n \times n$ matrices $\rightsquigarrow$ top-right coefficient.
- $2 \times 2 \times 2$ tensors $\rightsquigarrow$ coefficient at $e_1 \otimes e_1 \otimes e_1$
- A binary form is monic when it is.

Remark

In each case, the function $h \in V^*$ is a highest weight vector.

Examples of upper bounds on the monic rank

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Theorem

- *The maximum monic rank of an* $n \times m$ *matrix equals* $\min(n, m)$.
- *The maximum monic rank of a symmetric* $n \times n$ *matrix equals* n.
- *The maximum monic rank of a trace-zero* $n \times n$ *matrix equals* n.
- *The maximum monic rank of a* $2 \times 2 \times 2$ *tensor equals* 3.

Conjecture (monic version)

Every homogeneous binary form of degree $d \cdot e$ *with leading coefficient* d *is the sum of* d d*-th powers of monic forms of degree* e.

Theorem

This is true for $e = 1$, $d = 1, 2$ *and* $(d, e) = (3, 2), (3, 3), (3, 4), (4, 2)$.

2 x 2 x 2 Tensors

$$V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \left\{ \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{array} \right) \,\middle|\, a_{ij}, b_{ij} \in \mathbb{C} \right\}$$

2 x 2 x 2 Tensors

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We have

$$X \cap H = \left\{ \left(\begin{array}{cc|cc} 1 & b & c & bc \\ a & ab & ac & abc \end{array} \right) \middle| \, a, b, c \in \mathbb{C} \right\}$$

2 x 2 x 2 Tensors

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and 3 commuting actions of $\mathbb{C}$ on V:

- $(v_1 \ v_2 \mid w_1 \ w_2) \rightsquigarrow (v_1 \ v_2 + \lambda v_1 \mid w_1 \ w_2 + \lambda w_1)$
- $\left(\begin{array}{c|c} r_1 & s_1 \\ r_2 & s_2 \end{array} \right) \rightsquigarrow \left(\begin{array}{c|c} r_1 & s_1 \\ r_2 + \lambda r_1 & s_2 + \lambda s_1 \end{array} \right)$
- $(A \mid B) \rightsquigarrow (A \mid B + \lambda A)$

$X \cap H + X \cap H = \ ???$

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$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right) = \left(\begin{array}{cc|cc} 1 & b & c & bc \\ a & ab & ac & abc \end{array}\right) + \left(\begin{array}{cc|cc} 1 & -b & -c & bc \\ -a & ab & ac & -abc \end{array}\right)$$

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$$= \left(\begin{array}{cc|cc} 2 & 0 & 0 & 2bc \\ 0 & 2ab & 2ac & 0 \end{array}\right)$$

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$$
\Rightarrow X \cap H + X \cap H = \mathbb{C}^3 \cdot \left\{ \left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & 0 \end{array}\right) \;\middle|\; \begin{array}{l} \mu_1, \mu_2, \mu_3 \in \mathbb{C}, \\ \#\{i \mid \mu_i = 0\} \neq 1 \end{array} \right\}
$$

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Idea: Take a tensor

$$\left(\begin{array}{cc|cc} 3 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right),$$

modify it using our action and substract an element of $X \cap H$. Do this is such a way that the result is an element of $X \cap H + X \cap H$.

$(X \cap H + X \cap H) + X \cap H = 3H$

$$(a/3, b/3, c/3) \cdot \left(\begin{array}{cc|cc} 3 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right) - \left(\begin{array}{cc|cc} 1 & b & c & bc \\ a & ab & ac & abc \end{array}\right)$$

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$$=$$

$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 - 2bc/3 \\ 0 & \mu_3 - 2ab/3 & \mu_2 - 2ac/3 & \lambda + (a\mu_1 + b\mu_2 + c\mu_3)/3 - 8abc/9 \end{array}\right)$$

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We want:

- $\lambda + (a\mu_1 + b\mu_2 + c\mu_3)/3 - 8abc/9 = 0$
- $\mu_1 - 2bc/3 \neq 0$, $\mu_2 - 2ac/3 \neq 0$, $\mu_3 - 2ab/3 \neq 0$

$(X \cap H + X \cap H) + X \cap H = 3H$

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- $\mu_1 - 2bc/3 \neq 0$, $\mu_2 - 2ac/3 \neq 0$, $\mu_3 - 2ab/3 \neq 0$

This is doable unless $\lambda = \mu_1 = \mu_2 = \mu_3 = 0$. (But that case is easy.)

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Shapiro's Conjecture (Monic Version)

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We want to show that

$$\begin{array}{rcl} \prod_{i=1}^{d} \{\ell \in \mathbb{C}[x,y]_{(e)} \text{ monic}\} & \to & \mathbb{C}[x,y]_{(de)} \\ (\ell_1, \ldots, \ell_d) & \mapsto & \ell_1^d + \cdots + \ell_d^d \end{array}$$

is a finite morphism.

Shapiro's Conjecture (Monic Version)

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We want to show that

$$\begin{aligned} \prod_{i=1}^{d}\{\ell \in \mathbb{C}[x,y]_{(e)} \text{ monic}\} &\rightarrow \mathbb{C}[x,y]_{(de)} \\ (\ell_1,\dots,\ell_d) &\mapsto \ell_1^d + \cdots + \ell_d^d \end{aligned}$$

is a finite morphism.This is implied by:

Conjecture

The only solution of the equation

$$dx^{de} = \sum_{i=1}^{d} \left(x^e + c_{i1}x^{e-1}y + \cdots + c_{ie}y^e\right)^d$$

is $(c_{ij})_{ij} = 0$.

Reduction to a Gröbner basis computation

Assume that $(c_{ij})_{ij}$ satisfies

$$dx^{de} = \sum_{i=1}^{d} \left(x^e + c_{i1}x^{e-1}y + \cdots + c_{ie}y^e\right)^d$$

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Case 1

We have $c_{ie} = 0$ for all i. Devide by x^d.

$\rightsquigarrow$ This replaces e by $e-1$.

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Case 2

After permuting summands and scaling y, we get $c_{1e} = 1$.

$\leadsto$ A Gröbner basis can contradict this.

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The computation finished for $(d,e) = (3,2), (3,3), (3,4), (4,2)$.

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Thank you for your attention!

References

- Bik, Draisma, Oneto, Ventura, *The monic rank*, preprint.
- Lundqvist, Oneto, Reznick, Shapiro, *On generic and maximal k-ranks of binary forms*, Journal of Pure and Applied Algebra, 2018.