## Tropical Linear Algebra

Jorge Alberto Olarte

February 18, 2019

Joint work with Alex Fink, Benjamin Schröter and Marta Panizzut

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## Classical linear spaces

Let $\mathbb{K}$ be any field. A $d$-dimensional linear subspace $L \subseteq \mathbb{K}^{n}$ can be given in several forms:
(1) As a span of vectors $d$ vectors $v_{1}, \ldots, v_{d} \in \mathbb{K}^{d}$. This can be represented by a matrix $A \in \mathbb{K}^{d \times n}$ where the rows are given by the vectors $v_{1}, \ldots, v_{d}$.

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(2) As the solution set of $n-d$ linear equations. This can be represented by a matrix $A^{\perp} \in \mathbb{K}^{(n-d) \times n}$ where rows give the coefficients of the linear equations.

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(2) As the solution set of $n-d$ linear equations. This can be represented by a matrix $A^{\perp} \in \mathbb{K}^{(n-d) \times n}$ where rows give the coefficients of the linear equations.
(3) By its Plücker coordinates.

## Classical Plücker coordinates

For any $B \in\binom{[n]}{d}$ (a subset of size $d$ of $\{1, \ldots, n\}$ ) let $A_{B} \in \mathbb{K}^{d \times d}$ be the submatrix of $A$ consisting of the columns indexed by $B$.

## Definition

The Plücker coordinates of $L$ consist in a vector in $\mathbb{P K K}\binom{n}{d_{0}}^{-1}$ whose entries are the maximal minors $\left[\operatorname{det}\left(A_{B}\right)\right]_{B \in\binom{[n]}{d}}$ of any matrix $A$ whose rows form a basis of $L$. This does not depend on the choice of basis.

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A vector $W \in \mathbb{P K}\left(\begin{array}{l}\binom{n}{d}-1\end{array}\right.$ is the vector of Plücker coordinates of a linear space if and only if it satisfies the Plücker relations:

$$
\forall S \in\binom{[n]}{d-1}, \quad \forall T \in\binom{[n]}{d+1}, \quad \sum_{i \in T \backslash S} W_{T \backslash i} \cdot W_{S \cup i}=0
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The linear space $L$ can be recovered from the Plücker coordinates:

$$
L=\left\{x \in \mathbb{K}^{n} \left\lvert\, \forall T \in\binom{[n]}{d+1}\right., \sum_{i \in T} W_{T \backslash i} \cdot x_{i}=0\right\}
$$

## The tropical semiring

The tropical semiring is $\mathbb{T}=(\mathbb{R} \cup\{\infty\}, \oplus, \odot, \infty, 0)$ where

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\begin{aligned}
& a \oplus b=\min (a, b) \\
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The tropical projective space is $\mathbb{T}^{n-1}=\left(\mathbb{T}^{n} \backslash\{(\infty, \ldots, \infty)\}\right) / \sim$ where $\left(x_{1}, \ldots, x_{n}\right) \sim\left(y_{1}, \ldots, y_{n}\right)$ if and only if there exists $\lambda \in \mathbb{T}$ such that

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A tropical polynomial is of the form

$$
\begin{aligned}
f\left(x_{1}, \ldots, x_{n}\right) & =\bigoplus_{\alpha \in \Delta} c_{\alpha} \odot x^{\odot \alpha} \\
& =\min _{\alpha \in \Delta}\left(c_{\alpha}+\alpha_{1} \cdot x_{1}+\cdots+\alpha_{n} \cdot x_{n}\right)
\end{aligned}
$$

where the coefficients $c_{\alpha} \in \mathbb{T}$ are tropical numbers and $\Delta$ is a finite subset of $\mathbb{N}^{n}$. The tropical analogous of the 'zeros' of $f$ consists of al $x \in \mathbb{T}^{n}$ where the minimum in $f$ is either $\infty$ or achieved twice.

## Tropical linear spaces

## Definition

A valuated matroid $W$ is a vector in $\mathbb{P T} T_{\binom{n}{d}-1}$ that satisfies the 3 -term tropical Plücker relations: for every $S \in\binom{[n]}{d-2}$ and $i, j, k, I \in[n] \backslash S$, the minimum of

$$
\left(W_{s i j} \odot W_{s k l}\right) \oplus\left(W_{s i j} \odot W_{s k l}\right) \oplus\left(W_{S i j} \odot W_{s k l}\right)
$$

is achieved twice. The tropical linear space associated to $W$ is

$$
L:=\left\{x \in \mathbb{T}^{n} \left\lvert\, \forall T \in\binom{[n]}{d+1}\right. \text {, the minimum } \bigoplus_{i \in T} W_{T \backslash i} \cdot x_{i} \text { is achieved twice }\right\}
$$

Tropical linear spaces are polyhedral complexes.

## Matroid polytopes

## Definition

The hypersimplex $\Delta_{d, n}$ is the intersection of unit hypercube $[0,1]^{n}$ with the hyperplane $\left\{x_{1}+\cdots+x_{n}=d\right\}$. A matroid polytope $M$ is a subpolytope of $\Delta_{d, n}$ such that all of its edges are also edges of $\Delta_{d, n}$


The hypersimplex $\Delta_{2,4}$ is an octahedron.


This square pyramid is matroid polytope.


This symplex is NOT a matroid polytope.

## Regular subdivisions

## Definition

Given a polytope $P$ with vertex set $V$ and a height function $h: V \rightarrow \mathbb{R}$ the lower faces of $\operatorname{conv}\left(\left\{(v, h(v)) \in \mathbb{R}^{n+1} \mid v \in V\right\}\right)$ project onto $P$ to form a polyhedral subdivision Sub(h). Such subdivisions are called regular. This gives $\mathbb{R}^{V}$ a fan structure $\operatorname{Sec}(P)$ called the secondary fan of $P$, which consists of cones $\sigma(\mathcal{S})=\left\{h \in \mathbb{R}^{V} \mid \operatorname{Sub}(h)=\mathcal{S}\right\}$ for each regular subdivision $\mathcal{S}$.


## Matroid subdivisions

## Theorem (Speyer)

- A vector $W \in \mathbb{P T}^{\binom{n}{d}-1}$ is a valuated matroid if and only if the polytope $\operatorname{conv}\left(\left\{e_{B} \mid W_{B}<\infty\right\}\right)$ is a matroid polytope and the regular subdivision Sub(W) consists of only matroid polytopes.


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- The tropical linear space $L$ associated to $W$ is dual to the subcomplex of Sub(W) which consists of all polytopes of Sub(W) that are not contained in any of the coordinate hyperplanes $\left\{x_{i}=0\right\}$


## Example

The matroid subdivision of $W \in \mathbb{P} \mathbb{T}^{\binom{4}{2}-1}$ where $W_{B}=0$ for $B \in\binom{4}{2} \backslash\{34\}$ and $W_{34}=1$ consists of the two square pyramids and its faces:


## The Dressian

Given a matroid polytope $M$, the space of all valuated matroids with support in $M$ is called the $\operatorname{Dressian~} \operatorname{Dr}(M)$. It is a subfan of the secondary fan $\operatorname{Sec}(M)$.

## Theorem (O.-Panizzut-Schröter)

The fan structure of the $\operatorname{Dressian~} \operatorname{Dr}(M)$ given by 3-term Plücker relations is the same as the fan structure as a subfan of $\operatorname{Sec}(M)$. In other words, matroid subdivisions are determined by their 3-skeleta.

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## Corollary

Binary matroid polytopes are indecomposable, that is, matroid polytopes that do not admit non-trivial regular matroid subdivisions.

However there are non binary indecomposable matroids.

## Problem

Classify all indecomposable matroids.

## Tropical matrices and the Stiefel Map

## Definition

Given a tropical matrix $A \in \mathbb{T}^{d \times n}$, the tropical Stiefel map

$$
\begin{aligned}
\pi: \mathbb{T}^{d \times n} & \longrightarrow \mathbb{P T}^{\binom{n}{d}-1} \\
A & \mapsto\left[\operatorname{tdet}\left(A_{B}\right)\right]_{B \in\binom{[n]}{d}}
\end{aligned}
$$

maps $A$ to its tropical minors.
This is the tropical analog of taking the linear span of $d$ points. Not all tropical linear spaces arise this way:


## Transversal Matroids

There is a special kind of matroids that come from matchings in bipartite graphs. Such matroids are called transversal.

## Theorem (Fink-Rincón, Fink-O.)

A valuated matroid is in the image of the tropical Stiefel map if and only if all the facets of its matroid subdivision come from transversal matroids.

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We explicitly describe the fibres of the tropical Stiefel map.

## Theorem (Fink-O.)

Given a transversal valuated matroid $W$ its preimage $\pi^{-1}(W)$ consists of the $S_{d}$-orbit (by permuting rows) of a certain product of fans.

## Example

 $B \in\binom{4}{2} \backslash\{34\}$ and $W_{34}=1$.

Matrices in $\pi^{-1}(W)$ :

$$
\begin{array}{lll}
\left(\begin{array}{llll}
* & 0 & 0 & 0 \\
0 & 0 & * & 1
\end{array}\right) & \left(\begin{array}{llll}
0 & * & 0 & 0 \\
0 & 0 & * & 1
\end{array}\right) \\
\left(\begin{array}{llll}
* & 0 & 0 & 0 \\
0 & 0 & 1 & *
\end{array}\right) & \left(\begin{array}{llll}
0 & * & 0 & 0 \\
0 & 0 & 1 & *
\end{array}\right)
\end{array}
$$



## Thanks for your attention!

