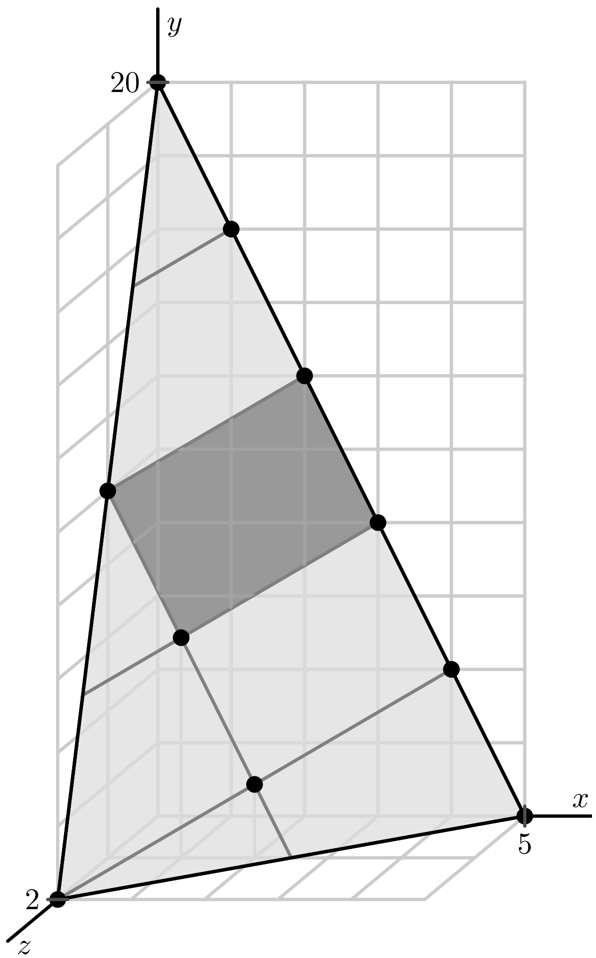
Discrete Volume Computations for Polyhedra

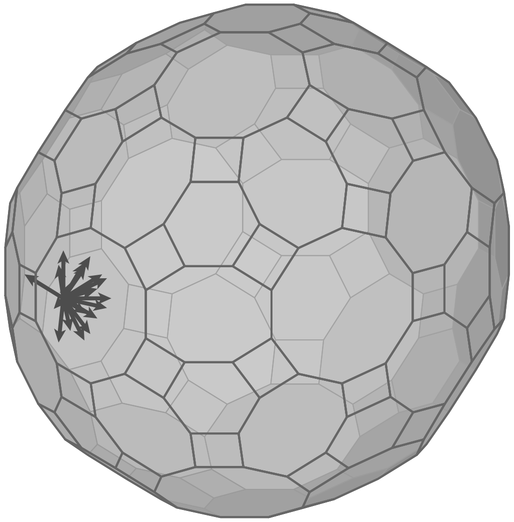


Matthias Beck
San Francisco State University
math.sfsu.edu/beck

Graduate Student Meeting
Applied Algebra & Combinatorics

Max-Planck-Institut für Mathematik in den Naturwissenschaften
MAX-PLANCK-GESELLSCHAFT

Themes

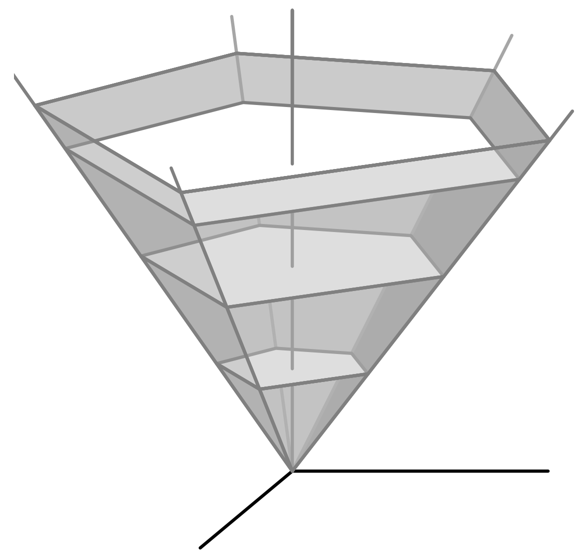


Combinatorial polynomials

Computation (complexity)

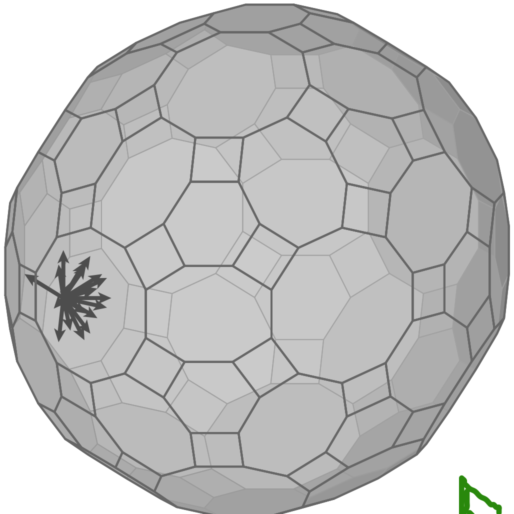
Generating functions

Combinatorial structures



Polyhedra

Themes



Nonlinear Algebra

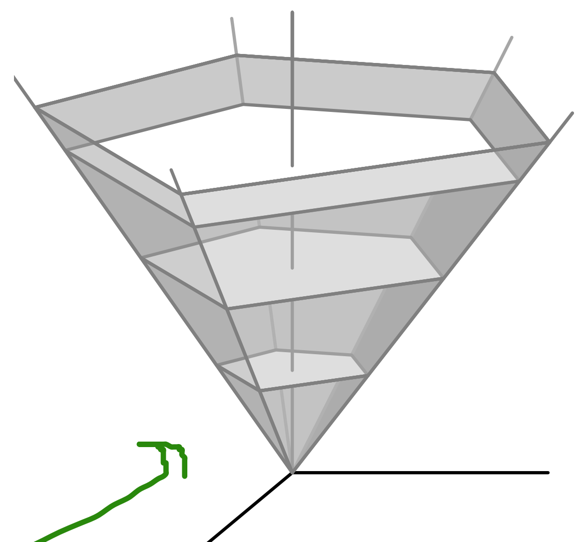
Combinatorial polynomials

Computation (complexity)

Generating functions

Combinatorial structures

Polyhedra



Linear Algebra

Motivation I: Birkhoff–von Neumann Polytope

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THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A037302 Normalized volume of Birkhoff polytope of n X n doubly-stochastic square matrices. If the volume is v(n), then a(n) = ((n-1)^2)! * v(n) / n^(n-1). 2

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028, 12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099, 5091038988117504946842559205930853037841762820367901333706255223000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS The Birkhoff polytope is an (n-1)^2-dimensional polytope in n^2-dimensional space; its vertices are the n! permutation matrices.
Is a(n) divisible by n^2 for all n>=4? - Dean Hickerson, Nov 27 2002

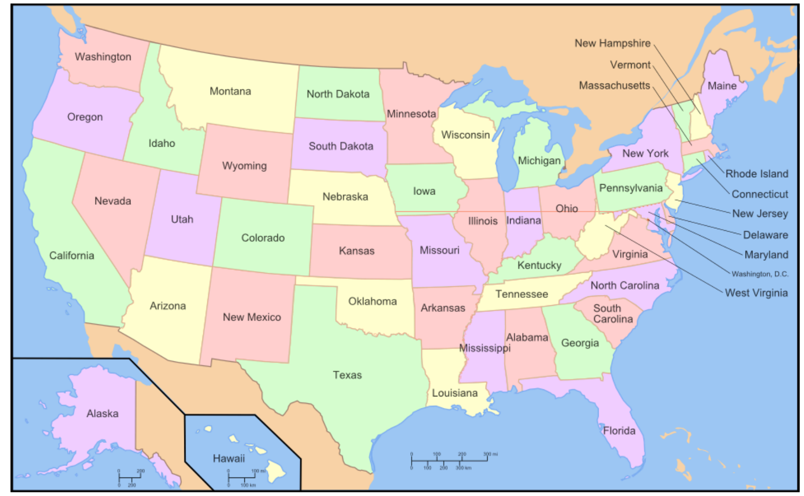
$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

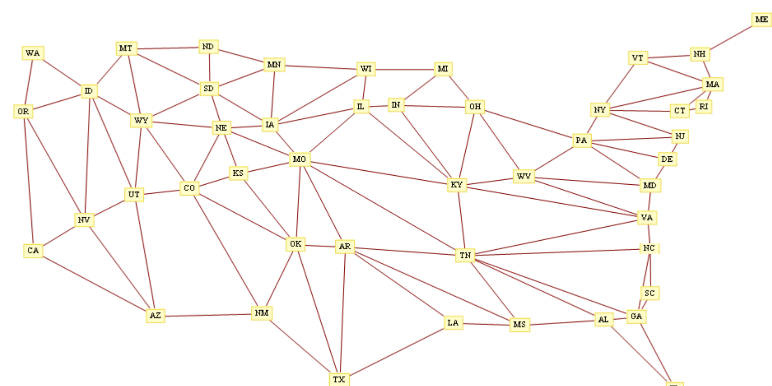
Motivation II: Polynomial Method 101

Theorem [Appel & Haken 1976]
The chromatic number of any planar graph is at most 4.

This theorem had been a conjecture (conceived by Guthrie when trying to color maps) for 124 years.

Birkhoff [1912] says:
Try polynomials!





[mathforum.org]

Four-Color Theorem Rephrased For a planar graph G, we have $\chi_G(4) > 0$, that is, 4 is not a root of the polynomial $\chi_G(k)$.

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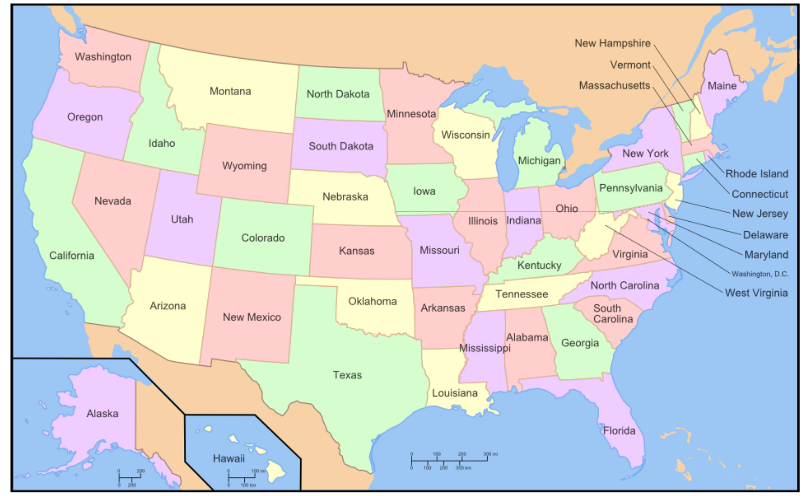

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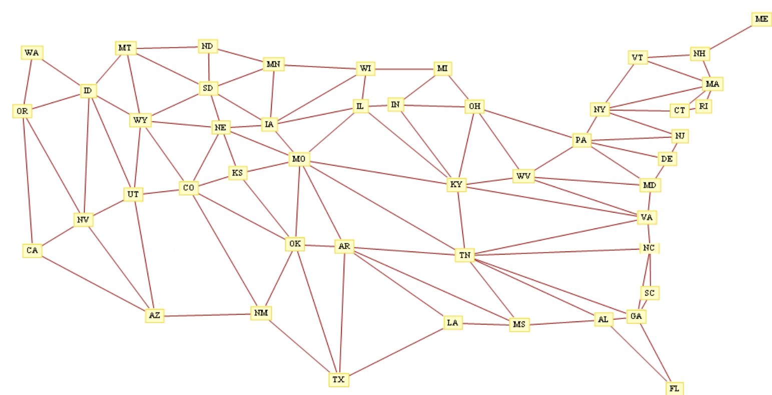

[mathforum.org]

Stanley [EC 1] says:
Try monomial algebras and generating functions!

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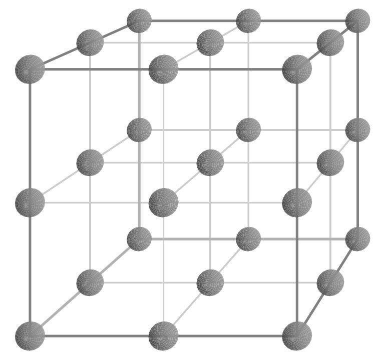
Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients



Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .

- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

- (count) $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$

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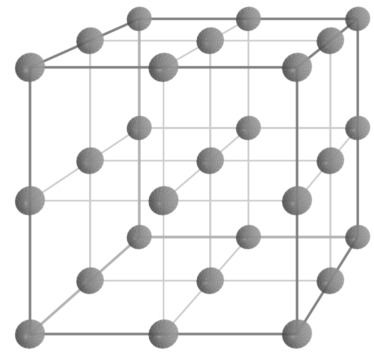

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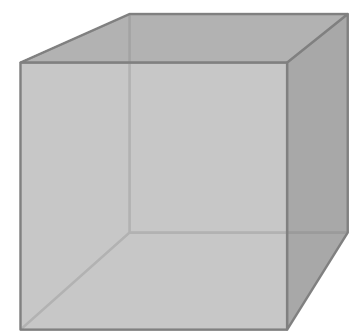

- (volume) $\displaystyle \mathrm{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left|\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right|$

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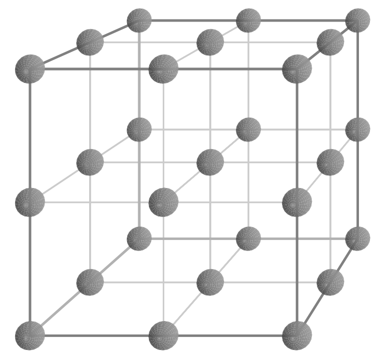

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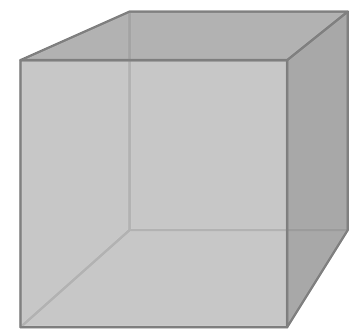

Ehrhart function $\displaystyle L_{\mathcal{P}}(t) := \left|\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right| = \left|t\mathcal{P} \cap \mathbb{Z}^d\right|$ for $t \in \mathbb{Z}_{>0}$

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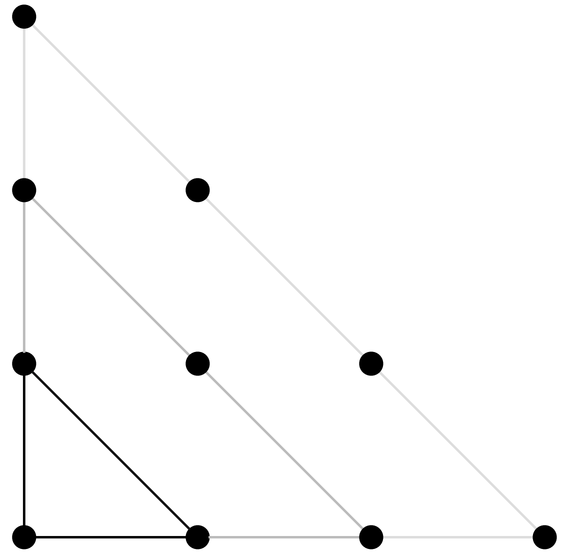
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.
- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.

A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$



Example 1:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{(x,y) \in \mathbb{R}^2_{\geq 0} : x + y \leq 1\right\}$$

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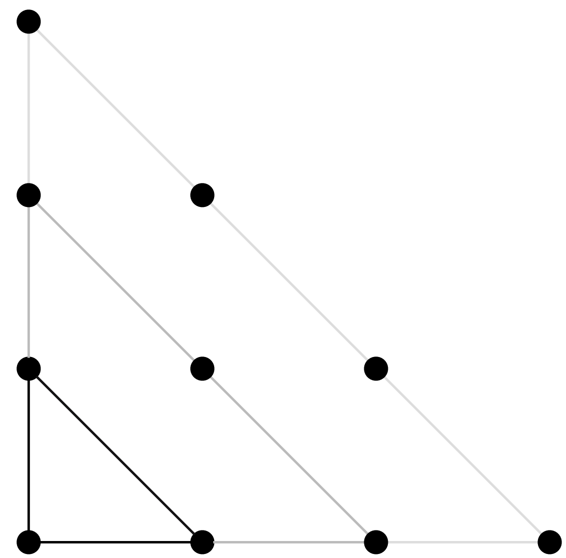

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Example 2:

$\square = [0,1]^d$ (the unit cube in $\mathbb{R}^d$)

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

Equivalently, $\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t$ is rational:

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h^*(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the Ehrhart h-vector $h^*(z)$ satisfies $h^*(0) = 1$ and $h^*(1) = (\dim \mathcal{P})! \operatorname{vol}(\mathcal{P})$.

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Seeming dichotomy: $\operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

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Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

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Equivalent descriptions of an Ehrhart polynomial:

- $L_{\mathcal{P}}(t) \ = \ c_d \, t^d + c_{d-1} \, t^{d-1} + \cdots + c_0$
- via roots of $L_{\mathcal{P}}(t)$
- $\mathrm{Ehr}_{\mathcal{P}}(z) \quad \longrightarrow \quad L_{\mathcal{P}}(t) \ = \ h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$

$h^*(z)$ is the binomial transform of $L_{\mathcal{P}}(t)$

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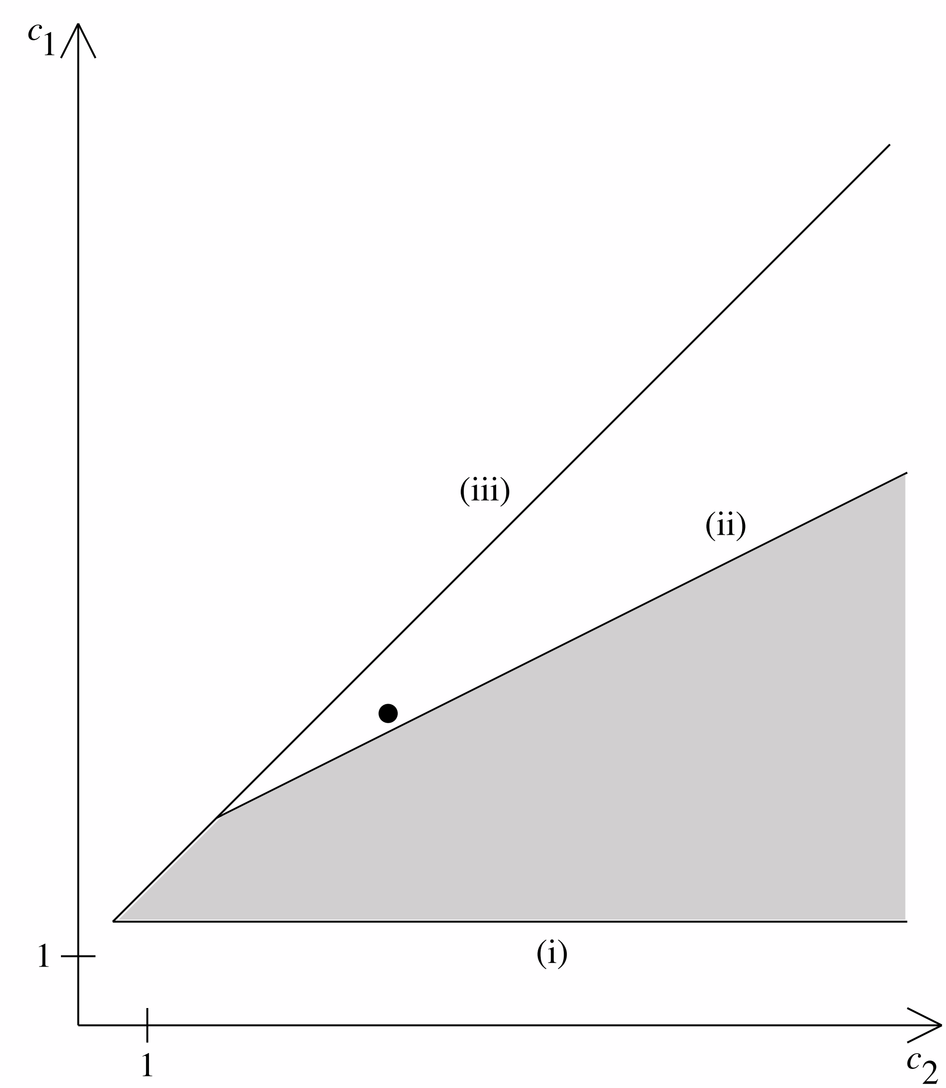
Open Problem Classify Ehrhart polynomials.

Two-dimensional Ehrhart Polynomials



Essentially due to Pick (1899) and Scott (1976)

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$$\longrightarrow \qquad L_{\mathcal{P}}(t) \; = \; h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$$

Theorem (Macdonald 1971) $(-1)^d L_{\mathcal{P}}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$. Equivalently,

$$L_{\mathcal{P}^\circ}(t) \; = \; h_d^* \binom{t+d-1}{d} + h_{d-1}^* \binom{t+d-2}{d} + \cdots + h_0^* \binom{t-1}{d}$$

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

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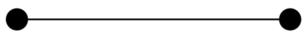
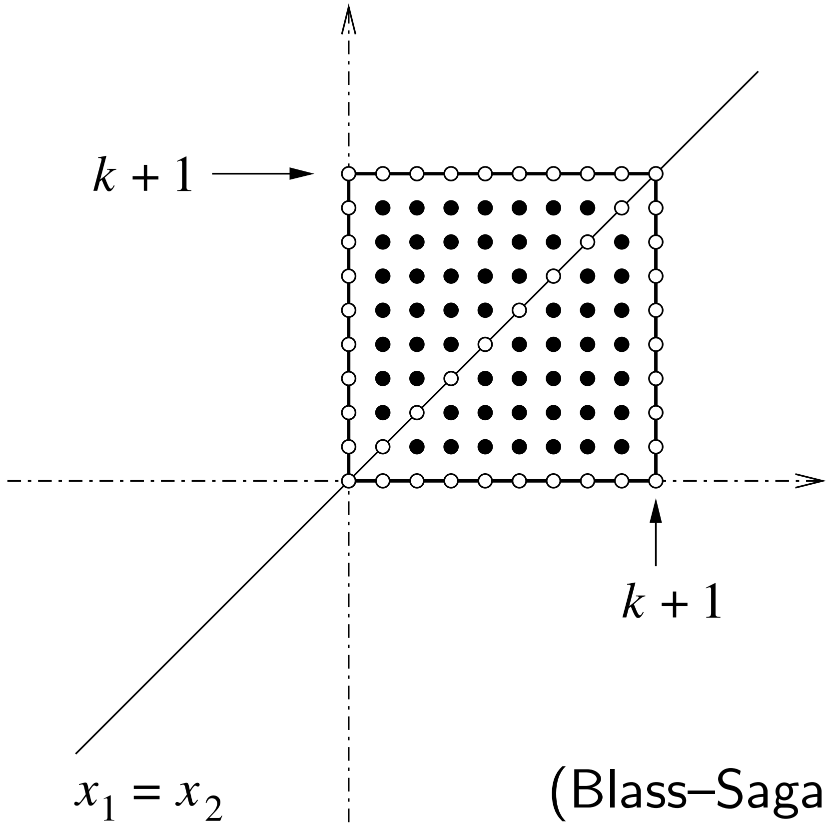
Corollary If $h_{d+1-k}^* > 0$ then $k\mathcal{P}^\circ$ contains an integer point.

Interlude: Graph Coloring a la Ehrhart

$$\chi_{K_2}(k) = 2\binom{k}{2} \ \ldots$$



K_2



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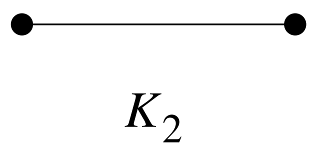

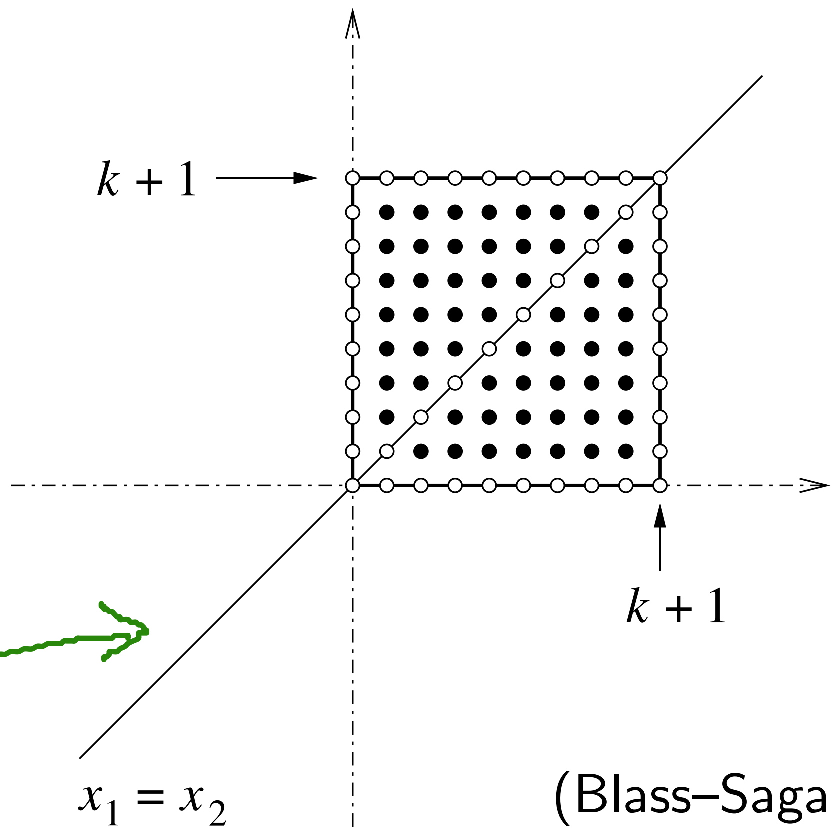

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$\boxed{\text{Nonequation Algebra}}$

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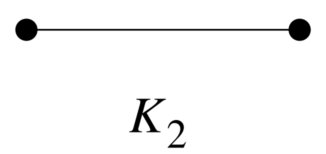

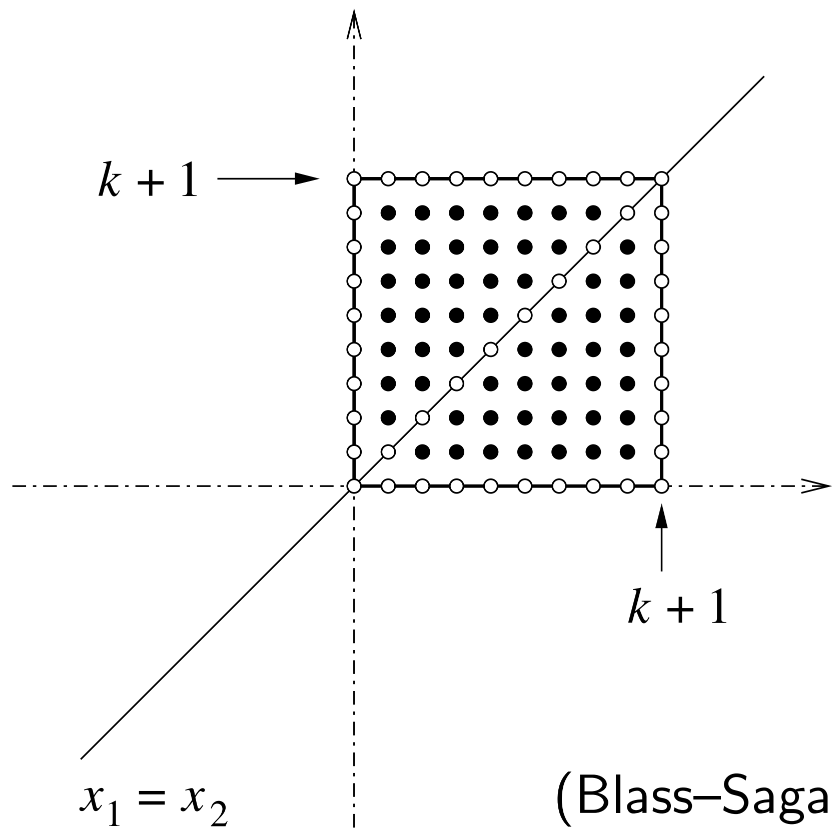

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Similarly, for any given graph G on d nodes, we can write

$$\chi_G(k) \;=\; \chi_0^*\binom{k+d}{d} + \chi_1^*\binom{k+d-1}{d} + \cdots + \chi_d^*\binom{k}{d}$$

for some (meaningful) nonnegative integers $\chi_0^*, \ldots, \chi_d^*$

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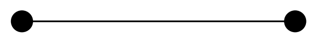

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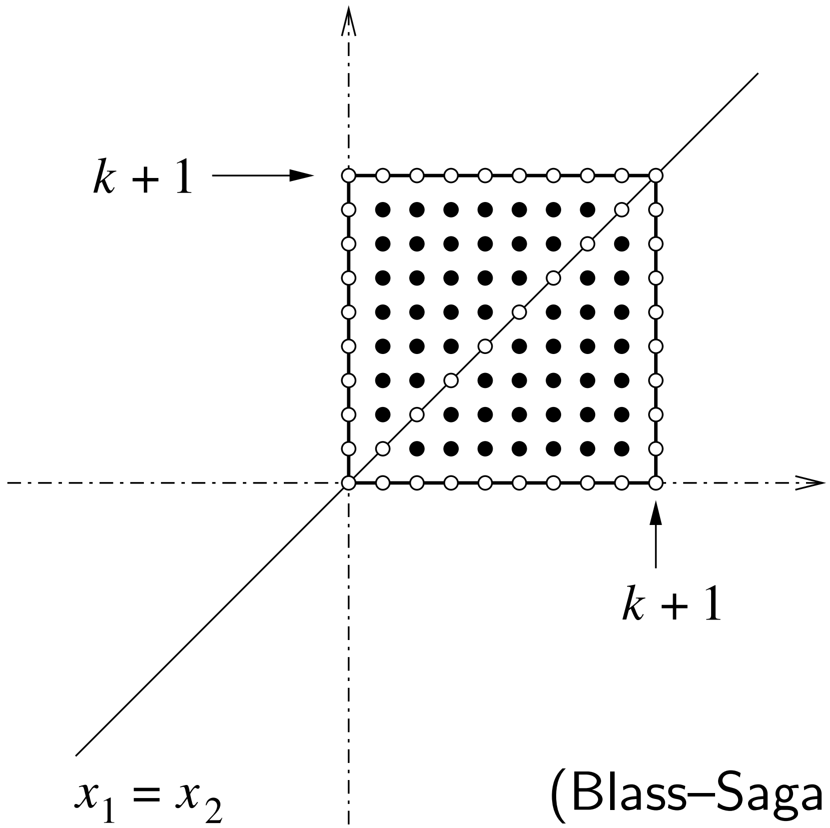

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Half-Open Problem Prove that $\chi_j^* > 0$ for some $0 \le j \le 4$ if G is planar.

Ehrhart h^* Positivity Refined

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} \left| t\mathcal{P} \cap \mathbb{Z}^d \right| z^t = \frac{h^*(z)}{(1-z)^{d+1}}$$

Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

Theorem (Betke–McMullen 1985, Stapledon 2009) If $h_d^* > 0$ then

$$h^*(z) = a(z) + z\, b(z)$$

where $a(z) = z^d\, a(\frac{1}{z})$ and $b(z) = z^{d-1}\, b(\frac{1}{z})$ with nonnegative coefficients.

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where $a(z) = z^d\, a(\frac{1}{z})$ and $b(z) = z^{d-1}\, b(\frac{1}{z})$ with nonnegative coefficients.

Open Problem Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

More Binomial Transforms

Chromatic polynomial $\chi_G(k) = \chi_0^* \binom{k+d}{d} + \chi_1^* \binom{k+d-1}{d} + \cdots + \chi_d^* \binom{k}{d}$

$\longrightarrow$ binomial transform $\chi_G^*(z) := \chi_d^* \, z^d + \chi_{d-1}^* \, z^{d-1} + \cdots + \chi_0^*$

Theorem (MB–León 2019+) Let G be a graph on d vertices. Then there exist symmetric polynomials $a_G(z) = z^d a_G(\frac{1}{z})$ and $b_G(z) = z^{d-1} b_G(\frac{1}{z})$ with positive integer coefficients such that

$$\chi_G^*(z) \ = \ a_G(z) - b_G(z) \, .$$

Moreover, $a_0 \le a_1 \le a_j$ where $1 \le j \le d-1$, and $b_0 \le b_1 \le b_j$ where $1 \le j \le d-2$.

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Theorem (Hersh–Swartz 2008) $\chi_{d-j}^* \ge \chi_j^*$ for $2 \le j \le \frac{d-1}{2}$

Similar results hold for flow polynomials of graphs (Breuer–Dall 2011).

Unimodal & Real-rooted Polynomials

The polynomial $h^*(z) = \sum_{j=0}^d h_j^* z^j$ is unimodal if for some $k \in \{0, 1, \ldots, d\}$

$$h_0^* \le h_1^* \le \cdots \le h_k^* \ge \cdots \ge h_d^*$$

Crucial Example $h^*(z)$ has only real roots

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Classic Example $\mathcal{P} = [0,1]^d$ comes with the Eulerian polynomial $h^*(z)$

Theorem (Schepers–Van Langenhoven 2013) $h^*(z)$ is unimodal for lattice parallelepipeds.

Theorem (MB–Jochemko–McCullough 2019) $h^*(z)$ is real rooted for lattice zonotopes.

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Conjectures $h^*(z)$ is unimodal/real-rooted for

- hypersimplices
- order polytopes
- alcoved polytopes
- lattice polytopes with unimodular triangulations
- IDP polytopes (integer decomposition property)

A Polynomial Ansatz to Antimagic Graph Labelings

An antimagic labeling of $G = (V, E)$ is an assignment $E \to \mathbb{Z}_{>0}$ such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

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Conjecture [Hartsfield & Ringel 1990] Every connected graph except K_2 has an antimagic labeling.

- [Alon et al 2004] connected graphs with minimum degree $\geq c \log |V|$
- [Bérczi et al 2017] connected regular graphs
- open for trees

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Idea Introduce a counting function: let $A^*_G(k)$ be the number of assignments of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$ and is distinct;
- the sum of the labels on all edges incident with a given node is unique.

Then G has an antimagic labeling if and only if $A^*_G(|E|) > 0$.

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Bad News The counting function $A^*_G(k)$ is in general not a polynomial:

$$A^*_{C_4}(k) = k^4 - \frac{22}{3}k^3 + 17k^2 - \frac{38}{3}k + \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$

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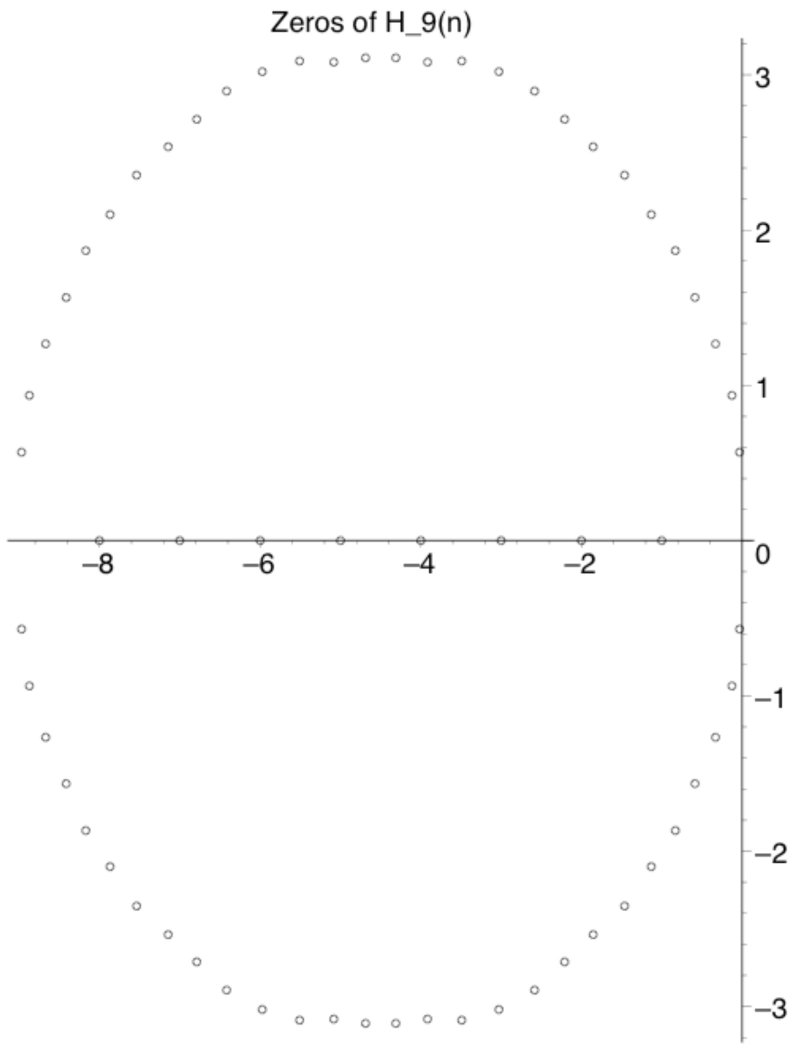
New Idea Introduce another counting function: let $A_G(k)$ be the number of assignments of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sum of the labels on all edges incident with a given node is unique.

Theorem (MB–Farahmand 2017) $A_G(k)$ is a quasipolynomial in k of period at most 2. If G minus its loops is bipartite then $A_G(k)$ is a polynomial.

Corollary For bipartite graphs, $A_G^*(|E|) > 0$.

One Last Picture: Birkhoff–von Neumann Roots



For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).