

# Algebraic Combinatorics in Geometric Complexity Theory

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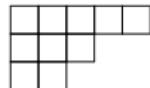
## Combinatorics and Representation Theory basics

**Symmetric group**  $S_n$ : Permutations  $\pi : [1..n] \rightarrow [1..n]$  under composition.

**Integer partitions**  $\lambda \vdash n$ :

$$\lambda = (\lambda_1, \dots, \lambda_\ell), \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0, \lambda_1 + \lambda_2 + \dots = n$$

**Young diagram** of  $\lambda$ :



Here  $\lambda = (5, 3, 2)$

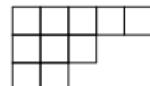
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**Representations of  $S_n$ :** group homomorphisms  $S_n \rightarrow GL(V)$ .

Example: if  $V = \mathbb{C}^3$ ,  $\pi \in S_3$ , set  $\pi(e_i) := e_{\pi_i}$  for  $i = 1..3$ , so e.g.  $231 \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

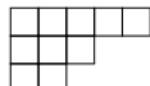
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**Irreducible decomposition:** minimal  $S_n$ -invariant subspaces  $V_i$ , so

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_k, \text{ e.g. } V = \underbrace{\mathbb{C}\langle e_1 + e_2 + e_3 \rangle}_{V_1} \oplus \underbrace{\mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle}_{V_2}$$

The **irreducible modules (representations)** (up to equivariant isomorphisms) of  $S_n$  are the **Specht modules**  $\mathbb{S}_\lambda$ , indexed by all  $\lambda \vdash n$ ,

e.g.  $V_1 \simeq \mathbb{S}_{\square\square}$  and  $V_2 \simeq \mathbb{S}_{\square\square\square}$

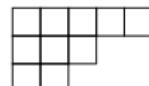
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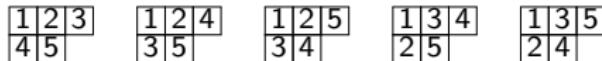
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**Basis for  $\mathbb{S}_\lambda$ :** Standard Young Tableaux of shape  $\lambda$ :  $\lambda = (3, 2)$

$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \hline \end{array}$
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# Young Tableaux and Schur functions

*Irreducible representations of the symmetric group  $S_n$ : Specht modules  $\mathbb{S}_\lambda$*



*Irreducible (polynomial) representations of the General Linear group  $GL_N(\mathbb{C})$ :*

**Weyl modules**  $V_\lambda$  (aka  $\mathcal{W}_\lambda$ ), indexed by highest weights  $\lambda$ ,  $\ell(\lambda) \leq N$ .

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**Schur functions:** characters of  $V_\lambda$

$$Tr_{V_\lambda}(diag(x_1, \dots, x_N)) = s_\lambda(x_1, \dots, x_N)$$

**Weyl's determinantal formula:**

$$s_\lambda(x_1, \dots, x_N) = \frac{\det \left[ x_i^{\lambda_j + N - j} \right]_{ij=1}^N}{\prod_{i < j} (x_i - x_j)}$$

**Semi-Standard Young tableaux** of shape  $\lambda$  :

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

$$\begin{array}{c|c|c|c|c|c} 1 & 1 & & & & \\ \hline 2 & 2 & & & & \\ \hline & & 1 & 1 & & \\ & & 3 & 3 & & \\ \hline & & 2 & 2 & & \\ & & 3 & 3 & & \\ \hline & & 1 & 1 & & \\ & & 2 & 3 & & \\ \hline & & 1 & 2 & & \\ & & 2 & 3 & & \\ \hline & & 1 & 2 & & \\ & & 3 & 3 & & \end{array}$$

## Products and compositions

Von Neumann et al, ca. 1934, representations of Lie groups:

$GL_N(\mathbb{C})$  acts on  $V_\lambda, V_\mu$  and their tensor product:

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$$

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(LR tableaux of shape  $(7, 4, 3)/(3, 1)$  and type  $(4, 3, 2)$ ).  $c_{(3,1)(4,3,2)}^{(7,4,3)} = 2$

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**Kronecker coefficients:**  $g(\lambda, \mu, \nu)$  – multiplicity of  $\mathbb{S}_\nu$  in  $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

E.g.:  $\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)} = \mathbb{S}_{(3)} \oplus \mathbb{S}_{(2,1)} \oplus \mathbb{S}_{(1,1,1)}$  and so  $g((2,1), (2,1), \nu) = 1$  for  $\nu = (3), (2,1), (1,1,1)$ .

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$$\text{Sym}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \bigoplus_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) V_\lambda \otimes V_\mu \otimes V_\nu$$

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**Plethysm coefficients** in  $GL$ -representation compositions:

$$GL_N \rightarrow GL(V_\mu) \rightarrow GL(V_\nu) \iff GL_N \rightarrow V_\nu[V_\mu] = \bigoplus_\lambda V_\lambda^{\oplus a_\lambda(\nu[\mu])}$$

# The Algebraic Combinatorics problems

## Problem (Murnaghan, 1938, then Stanley et al)

*Find a positive combinatorial interpretation for  $g(\lambda, \mu, \nu)$ , i.e. a family of combinatorial objects  $\mathcal{O}_{\lambda, \mu, \nu}$ , s.t.  $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda, \mu, \nu}$ . Alternatively, show that KRON (“Input:  $(\lambda, \mu, \nu)$ , output:  $g(\lambda, \mu, \nu)$ ”) is in  $\#P$ .*

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**Classical motivation:** (Littlewood–Richardson: for  $c_{\lambda, \mu}^{\nu}$ ,

$\mathcal{O}_{\lambda, \mu, \nu} = \{ \text{LR tableaux of shape } \nu/\mu, \text{ type } \lambda \}$  )

**Theorem [Murnaghan]** If  $|\lambda| + |\mu| = |\nu|$  and  $n > |\nu|$ , then

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## Modern motivation:

1. A positive combinatorial formula "  $\iff$  " Computing Kronecker coefficients is in  $\#P$  .
2. **Geometric Complexity Theory.**
3. Invariant Theory, moment polytopes [see Bürgisser, Christandl, Mülmuley, Walter, Oliveira, Garg, Wigderson etc]

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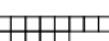
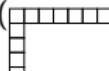
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**Results since then:**

Combinatorial formulas for  $g(\lambda, \mu, \nu)$ , when:

- $\mu$  and  $\nu$  are hooks (, [Remmel, 1989])
- $\nu = (n - k, k)$  () and  $\lambda_1 \geq 2k - 1$ , [Ballantine–Orellana, 2006]
- $\nu = (n - k, k)$ ,  $\lambda = (n - r, r)$  [Remmel–Whitehead, 1994; Blasiak–Mulmuley–Sohoni, 2013]
- $\nu = (n - k, 1^k)$  (, [Blasiak 2012, Blasiak-Liu 2014])
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova].

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### Bounds and positivity:

[Pak-P]:  $g(\lambda, \mu, \mu) \geq |\chi^\lambda(2\mu_1 - 1, 2\mu_2 - 3, \dots)|$  when  $\mu = \mu^T$ . Corollaries :

$g(\lambda, \mu, \mu) > c \frac{2^{\sqrt{2k}}}{k^{9/4}}$  for  $\lambda = (|\mu| - k, k)$ , and  $\text{diag}(\mu) \geq \sqrt{k}$ .

[Saxl conjecture]: For every  $n > 9$  there exists a self-conjugate partition  $\lambda \vdash n$ , s.t.  $g(\lambda, \lambda, \mu) > 0$  for all  $\mu \vdash n$ . When  $n = \binom{m+1}{2}$ , then  $\lambda = (m, m-1, \dots, 1)$ . [Partial results: Pak-P-Vallejo, Ikenmeyer, Luo-Sellke]

### Complexity results:

[Bürgisser-Ikenmeyer]: KRON is in GapP.

( Littlewood-Richardson, i.e. KRON's special case, is  $\#P$  -complete )

[Pak-P]: If  $\nu$  is a hook, then KronPositivity is in P. If  $\lambda, \mu, \nu$  have fixed length there exists a linear time algorithm for deciding  $g(\lambda, \mu, \nu) > 0$ .

[Ikenmeyer-Mulmuley-Walter]: KronPositivity is NP -hard.

[Bürgisser-Christandl-Mulmuley-Walter]: membership in the moment polytope is NP and coNP .

## Basic properties and formulas

From representation theory:

$$g(\lambda, \lambda, (n)) = g(\lambda, \lambda', (1^n)) = 1$$

Semigroup property: If  $\alpha, \beta, \gamma, \lambda, \mu, \nu$  are such that  $g(\alpha, \beta, \gamma) > 0$  and  $g(\lambda, \mu, \nu) > 0$  then  $g(\alpha + \lambda, \beta + \mu, \gamma + \nu) \geq \max\{g(\alpha, \beta, \gamma), g(\lambda, \mu, \nu)\}$

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Via Schur functions  $s_\lambda$ :

$$s_\lambda(x) = \sum_{T: SSYT, \text{sh}(T)=\lambda} x^T$$

$$s_\lambda\left[\underbrace{x \cdot y}_{x_1y_1, x_1y_2, \dots, x_2y_1, \dots}\right] = \sum_{\mu, \nu} g(\lambda, \mu, \nu) s_\mu(x) s_\nu(y)$$

Triple Cauchy identity:

$$\prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z)$$

A GAPP formula via Contingency Arrays: (in [Christandl-Doran-Walter, Pak-Panova])

$$g(\alpha, \beta, \gamma) = \sum_{\sigma^1, \sigma^2, \sigma^3 \in S_\ell} \text{sgn}(\sigma^1 \sigma^2 \sigma^3) CA(\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3),$$

$CA(u, v, w)$  = is # of  $\ell \times \ell \times \ell$  contingency arrays  $[A_{i,j,k}] \in \mathbb{N}^{k \times k \times k}$ :

$$\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_j, \quad \sum_{i,j} A_{i,j,k} = w_k$$

“Example”: when  $\nu = (n-k, k)$  – two rows

$\ell(\nu) = 2$ :

$$\begin{aligned} g(\lambda, \mu, \nu) &= \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \nu_i - i + \sigma_i, i=1,2} c_{\alpha^1 \alpha^2}^{\lambda} c_{\alpha^1 \alpha^2}^{\mu} \\ &= \underbrace{\sum_{\alpha \vdash k, \beta \vdash n-k} c_{\alpha \beta}^{\lambda} c_{\alpha \beta}^{\mu}}_{a_k(\lambda, \mu)} - \underbrace{\sum_{\alpha \vdash k-1, \beta \vdash n-k+1} c_{\alpha \beta}^{\lambda} c_{\alpha \beta}^{\mu}}_{a_{k-1}(\lambda, \mu)} \end{aligned}$$

### Corollary (Pak-P, Vallejo)

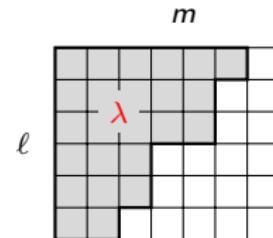
The sequence  $a_0(\lambda, \mu), a_1(\lambda, \mu), \dots, a_n(\lambda, \mu)$  is unimodal for all  $\lambda, \mu \vdash n$ , i.e.

$$a_0(\lambda, \mu) \leq a_1(\lambda, \mu) \leq \dots \leq a_{\lfloor n/2 \rfloor}(\lambda, \mu) \geq \dots \geq a_n(\lambda, \mu).$$

When  $\nu = (n - k, k)$  – two rows

$$p_n(\ell, m) = \#\{\lambda \vdash n, \ell(\lambda) \leq \ell, \lambda_1 \leq m\}$$

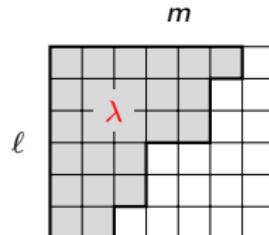
$$\sum_{n \geq 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1 - q^{m+i}}{1 - q^i} = \binom{m + \ell}{m}_q$$



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$$\sum_{n \geq 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1 - q^{m+i}}{1 - q^i} = \binom{m+\ell}{m}_q$$



Theorem (Sylvester 1878, Cayley's conjecture 1856)

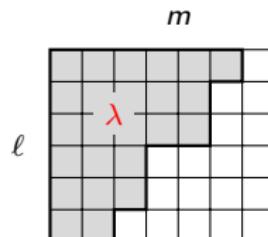
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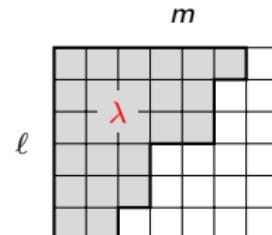
*"I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power."*

J.J. Sylvester, 1878.

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**Proof via Kronecker:** [Pak-P]

$$0 \leq g(\lambda, \mu, \nu) = \underbrace{\sum_{\alpha \vdash k, \beta \vdash n-k} c_{\alpha\beta}^\lambda c_{\alpha\beta}^\mu}_{a_k(\lambda, \mu)} - \underbrace{\sum_{\alpha \vdash k-1, \beta \vdash n-k+1} c_{\alpha\beta}^\lambda c_{\alpha\beta}^\mu}_{a_{k-1}(\lambda, \mu)}$$

$$a_k(m^\ell, m^\ell) = \sum_{\alpha \vdash k, \beta \vdash m\ell-k} \mathbb{1}(\beta_i = m - \alpha_{\ell+1-i}, i = 1 \dots \ell) = p_k(\ell, m)$$

+Corollary –  $a_k(\lambda, \mu)$  unimodal

More corollaries: strict unimodality via semigroup property, exponential lower bounds via characters...

## (Boolean) Complexity

**Input:** string of  $n$  bits, i.e.  $\text{size}(\text{input}) = n$ .

**Decision problems:**

Is there an object, s.t.... ?

P = solution can be found in time Poly(n)

NP = solution can be verified in Poly(n)  
(polynomial witness)

NP -Complete = in NP , and every NP problem can be reduced to it poly time;

**Counting problems:**

Compute  $F(\text{input}) = ?$

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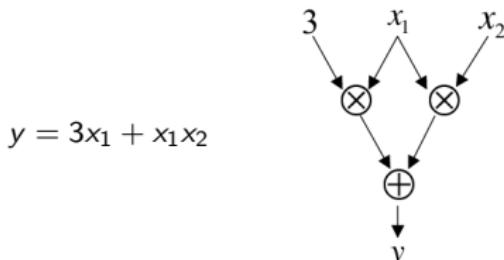
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An approach [Mulmuley, Sohoni]: **Geometric Complexity Theory**

## VP vs VNP: determinant vs permanent

### Arithmetic Circuits:



Polynomials  $f_n \in \mathbb{F}[X_1, \dots, X_n]$ . Circuit – nodes are  $+$ ,  $\times$  gates, input –  $X_1, \dots, X_n$  and constants from  $\mathbb{F}$ .

#### Class VP (Valliant's P):

polynomials that can be computed with  $\text{poly}(n)$  large circuit (size of the associated graph).

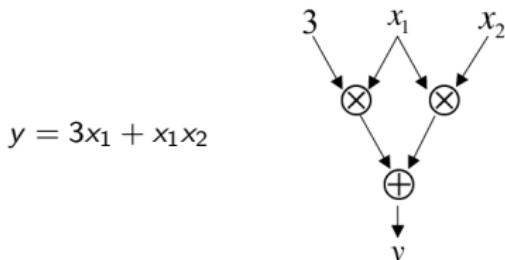
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**Theorem**[Bürgisser]: If  $\text{VP} = \text{VNP}$ , then  $\text{P} = \text{NP}$  if  $\mathbb{F}$  - finite or the Generalized Riemann Hypothesis holds.

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**Universality of the determinant [Cohn, Valiant]:**

For every polynomial  $p(X)$  there exists some  $n$  s.t.

$$p(X) = \det(A),$$

where  $A = [\ell_{i,j}(X)]_{i,j=1}^n$  with  $\ell_{i,j}(X) \in \{a_0 + a_1 X_1 + \dots + a_k X_k \mid a_i \in \mathbb{F}\}$ .

The smallest  $n$  possible is the *determinantal complexity*  $\text{dc}(p)$ .

**Example:**  $p = x_1^2 + x_1 x_2 + x_2 x_3 + 2x_1$ , then

$$p = \det \begin{bmatrix} x_1 + 2 & x_2 \\ -x_3 + 2 & x_1 + x_2 \end{bmatrix}, \quad \text{dc}(p) = 2$$

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Known:  $\text{dc}(\text{per}_m) \leq 2^m - 1$  (Grenet 2011),  $\text{dc}(\text{per}_m) \geq \frac{m^2}{2}$  (Mignon, Ressaire, 2004).

Ryser's formula:  $\text{per}_m(X) = (-1)^m \sum_{S \subset [1..m]} (-1)^{|S|} \prod_{i=1}^m (\sum_{j \in S} X_{i,j})$

# Geometric Complexity Theory

$GL_N$  action on polynomials:

$A \in GL_N(\mathbb{C})$ ,  $v := (X_1, \dots, X_N)$ ,  $f \in \mathbb{C}[X_1, \dots, X_N]$ ,  
then  $A.f = f(A^{-1}v)$  (replaces variables with linear forms)

$GL_{n^2}\det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$  – **determinant orbit**.

$\Omega_n := \overline{GL_{n^2}\det_n}$  – **determinant orbit closure**.

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$\max\{n : \text{per}_m^n \notin \overline{GL_{n^2}\det_n}\} (\leq \text{dc}(\text{per}_m))$  grows superpolynomially.

$$\text{per}_m^n \in \overline{GL_{n^2}\det_n} \iff \underbrace{\overline{GL_{n^2}\text{per}_m^n}}_{=: \Gamma_m^n} \subseteq \overline{\overbrace{GL_{n^2}\det_n}^{\Omega_n}}.$$

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Exploit the symmetry! Coordinate rings as  $\text{GL}_{n^2}$  representations:

$$\mathbb{C}[\overline{\text{GL}_{n^2} \det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[\overline{\text{GL}_{n^2} \text{per}_m^n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

**Definition (Representation theoretic obstruction)**

If  $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$ , then  $\lambda$  is a **representation theoretic obstruction**. Its existence shows  $\overline{\text{GL}_{n^2} \text{per}_m^n} \not\subseteq \overline{\text{GL}_{n^2} \det_n}$  and so  $\text{dc}(\text{per}_m) > n$  !

## (Non)existence of obstructions

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### Theorem (Bürgisser-Ikenmeyer-P(FOCS'16, JAMS'18))

*This Conjecture is false. There are no such occurrence obstructions.*

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**Theorem (Ikenmeyer-P (FOCS'16, Adv.Math.'17))**

Let  $n > 3m^4$ ,  $\lambda \vdash nd$ . If  $g(\lambda, n^d, n^d) = 0$  (so  $\text{mult}_\lambda \mathbb{C}[GL_{n^2}\det_n] = 0$ ), then  $\text{mult}_\lambda (\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}\text{per}_m]) = 0$ .

**Theorem (Ikenmeyer-P, FOCS'16, Adv.Math.'17)**

For any  $\rho$ , let  $n \geq |\rho|$ ,  $d \geq 2$ ,  $\lambda := (nd - |\rho|, \rho)$ . Then  $g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n])$ .



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**Conjecture (Mullmuley and Sohoni 2001)**

For all  $c \in \mathbb{N}_{\geq 1}$ , for infinitely many  $m$ , there exists a partition  $\lambda$  occurring in  $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} \text{per}_m}]$  but not in  $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$ , where  $n = m^c$ .

**Theorem (Bürgisser-Ikenmeyer-P (FOCS'16, JAMS'18))**

Let  $n, d, m$  be positive integers with  $n \geq m^{25}$  and  $\lambda \vdash nd$ . If  $\lambda$  occurs in  $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} \text{per}_m}]$ , then  $\lambda$  also occurs in  $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$ . In particular, the Conjecture is false, there are no “occurrence obstructions”.

# No occurrence obstructions I: positive Kroneckers

## Theorem (Ikenmeyer-P)

*Let  $n > 3m^4$ ,  $\lambda \vdash nd$ . If  $g(\lambda, n \times d, n \times d) = 0$ , then  
 $\text{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m} \text{per}_m]) = 0$ .*

### Proof:

$$\bar{\lambda} := (\lambda_2, \lambda_3, \dots) \vdash |\lambda| - \lambda_1$$

## Theorem (Kadish-Landsberg)

*If  $\text{mult}_\lambda \mathbb{C}[GL_{n^2} X_{11}^{n-m} \text{per}_m] > 0$ , then  $|\bar{\lambda}| \leq md$  and  $\ell(\lambda) \leq m^2$ .*

## Theorem (Degree lower bound, [IP])

*If  $|\bar{\lambda}| \leq md$  with  $a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$ , then  $d > \frac{n}{m}$ .*

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## Theorem (Kronecker positivity, [IP] )

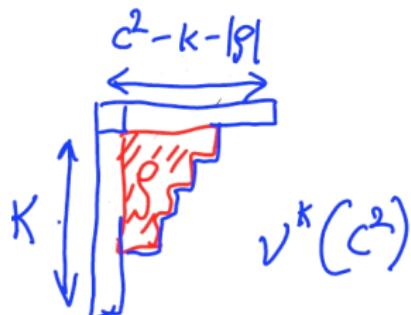
Let  $\lambda \vdash dn$ . Let  $\mathcal{X} := \{(1), (2 \times 1), (4 \times 1), (6 \times 1), (2, 1), (3, 1)\}$ .

(a) If  $\bar{\lambda} \in \mathcal{X}$ , then  $a_\lambda(d[n]) = 0$ .

(b) If  $\bar{\lambda} \notin \mathcal{X}$  and  $m \geq 3$  such that  $\ell(\lambda) \leq m^2$ ,  $|\bar{\lambda}| \leq md$ ,  $d > 3m^3$ , and  $n > 3m^4$ , then  
 $g(\lambda, n \times d, n \times d) > 0$ .

Kronecker positivity I: hook-like  $\lambda$ s

Proposition (Ikenmeyer-P)

*If there is an  $a$ , such that*

$g(\nu^k(a^2), a \times a, a \times a) > 0$  for all  $k$ , s.t.  $k \notin H^1(\rho)$  and  $a^2 - k \notin H^2(\rho)$  for some sets  $H^1(\rho), H^2(\rho) \subset [\ell, 2a + 1]$ ,  
*then  $g(\nu^k(b^2), b \times b, b \times b) > 0$  for all  $k$ , s.t.  $k \notin H^1(\rho)$  and  $b^2 - k \notin H^2(\rho)$  for all  $b \geq a$ .*

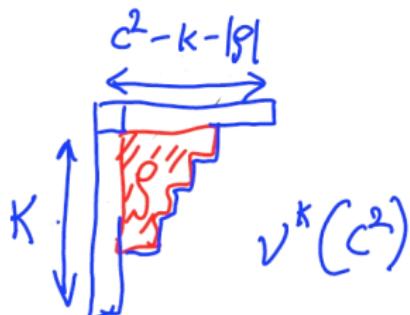
**Proof idea:**

Kronecker symmetries and semigroup properties:

Let  $P_c = \{k : g(\nu^k(c^2), c \times c, c \times c) > 0\}$ , we have**Claim:** Suppose that  $k \in P_c$ , then  $k, k + 2c + 1 \in P_{c+1}$ .

## Kronecker positivity I: hook-like $\lambda$ s

### Proposition (Ikenmeyer-P)



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**Claim:** Suppose that  $k \in P_c$ , then  $k, k + 2c + 1 \in P_{c+1}$ .

### Corollary

We have that  $g(\lambda, h \times w, h \times w) > 0$  for  $\lambda = (hw - j - |\rho|, 1^j + \rho)$  for most “small” partitions  $\rho$  and all but finitely many values of  $j$ .

## Kronecker positivity II: squares, and decompositions

### Theorem (Ikenmeyer-P)

Let  $\nu \notin \mathcal{X}$  and  $\ell = \max(\ell(\nu) + 1, 9)$ ,  $a > 3\ell^{3/2}$ ,  $b \geq 3\ell^2$  and  $|\nu| \leq ab/6$ . Then  $g(\nu(ab), a \times b, a \times b) > 0$ .

**Proof sketch:** decomposition + regrouping

$$\nu = \rho + \xi + \sum_{k=2}^{\ell} x_k((k-1) \times k) + \sum_{k=2}^{\ell} y_k((k-1) \times 2).$$

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### Crucial facts:

- $g(k \times k, k \times k, k \times k) > 0$  [Bessenrodt-Behns].
- Transpositions:  $g(\alpha, \beta, \gamma) = g(\alpha, \beta^T, \gamma^T)$  (with  $\beta = \gamma = w \times h$ )
- Hooks and exceptional cases:  $g(\lambda, h \times w, h \times w) > 0$  for all  $\lambda = (hw - j - |\rho|, 1^j + \rho)$  for  $|\rho| \leq 6$  and almost all  $j$ s.
- Semigroup property for positive triples:  

$$g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq \max(g(\alpha^1, \beta^1, \gamma^1), g(\alpha^2, \beta^2, \gamma^2)).$$

## Kronecker vs plethysm: inequality of multiplicities

**Stability[Manivel]:**  $g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d)$ , as  $n \rightarrow \infty$ .

$\text{St}^1(\rho) := \{(n, d) \mid g((nd - |\rho|, \rho), n \times d, n \times d)\} = a_\rho(d)$ .

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$$\implies \text{mult}_\lambda(\overline{\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]}) \geq a_\mu(d[m]) = a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$$

$$\implies \max_{f \in V_m} dc(f_{m,n}) > n \rightarrow \infty$$

Thank you!

## Algebraic Geometry

$$[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$$



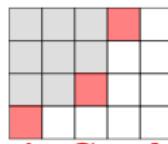
## Complexity Theory

P vs NP

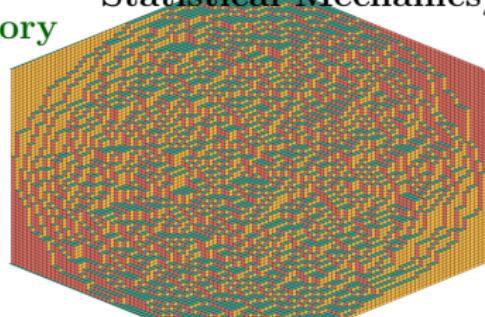
## Representation Theory



## Algebraic Combinatorics



## Statistical Mechanics/



## Probability

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

1   1	1   1	2   2	1   1	1   2	1   2
2   2	3   3	3   3	2   3	2   3	3   3