



A Geometric Description of Feasible Singular Values in Tree Tensor Formats

Sebastian Krämer

Feasibility and Compatibility —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

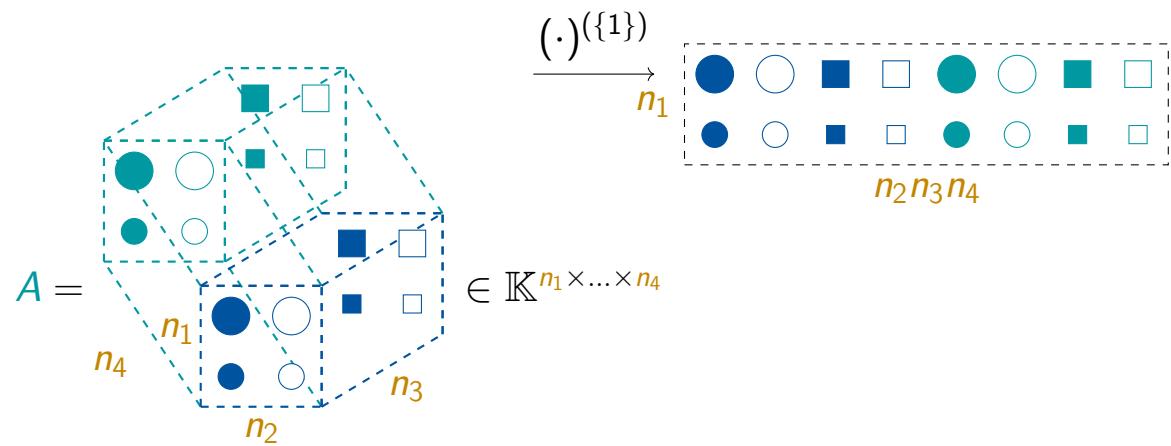
TT-Feasibility

Feasibility and Compatibility — Reshaping

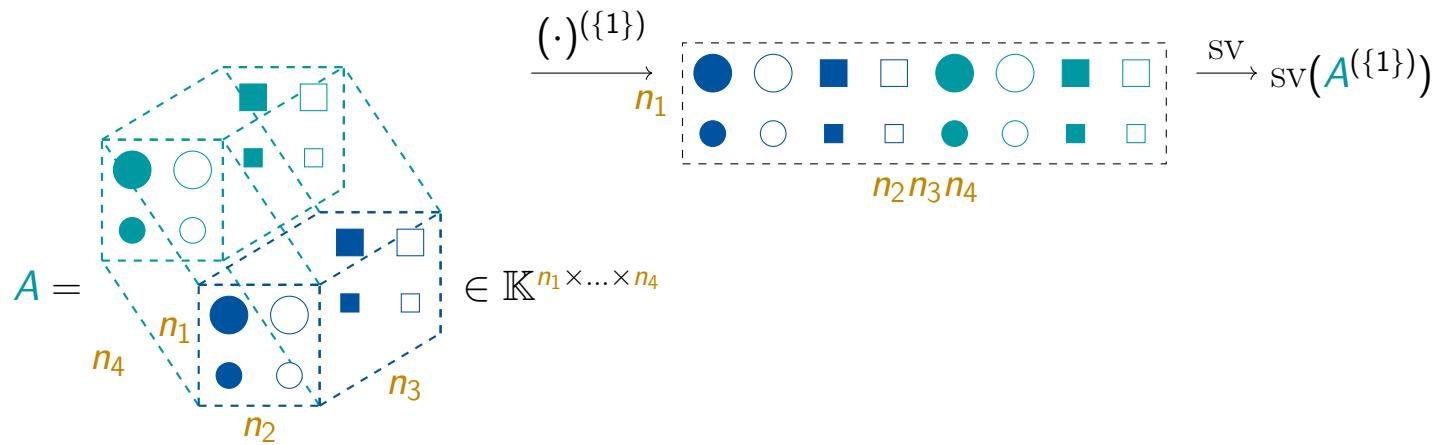
$$A = \begin{matrix} & & \\ & & \\ & & \\ & & \\ & & \\ & & \end{matrix} \in \mathbb{K}^{n_1 \times n_2 \times n_3}$$

Feasibility and Compatibility — Reshaping

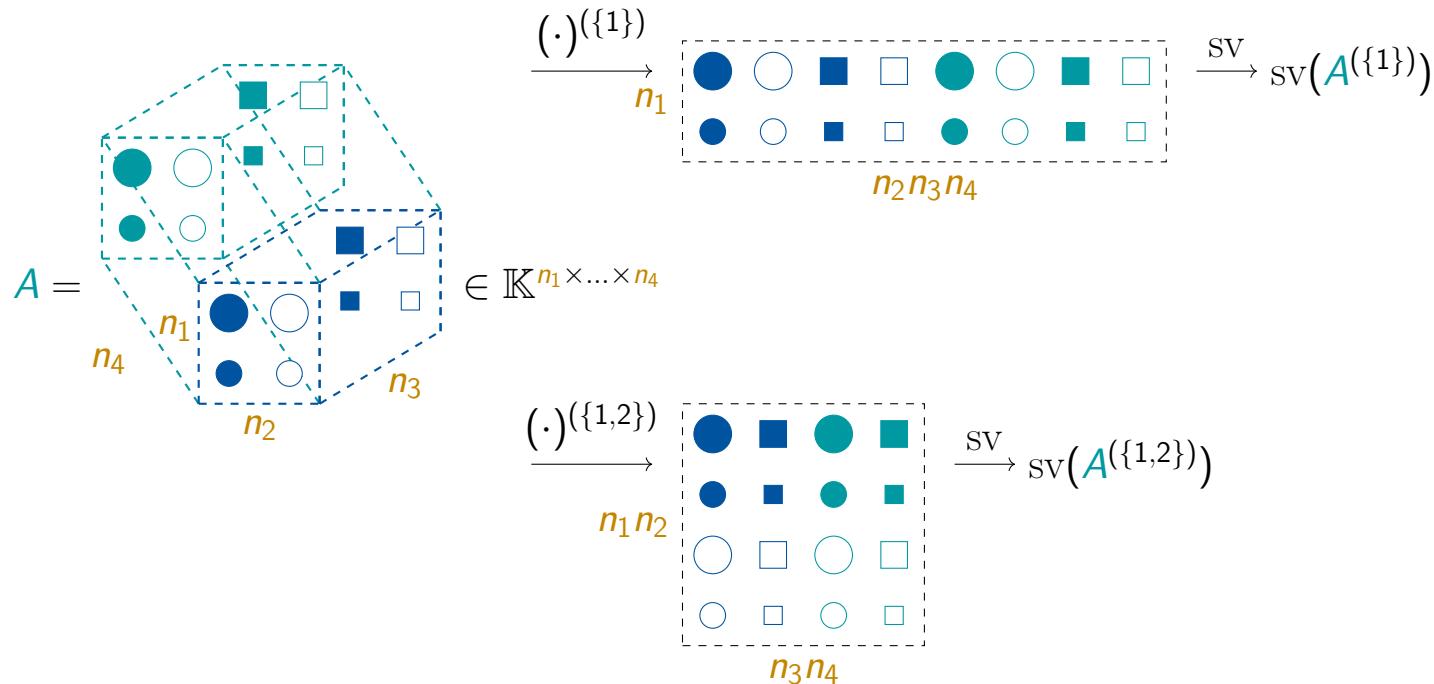
Feasibility and Compatibility — Reshaping



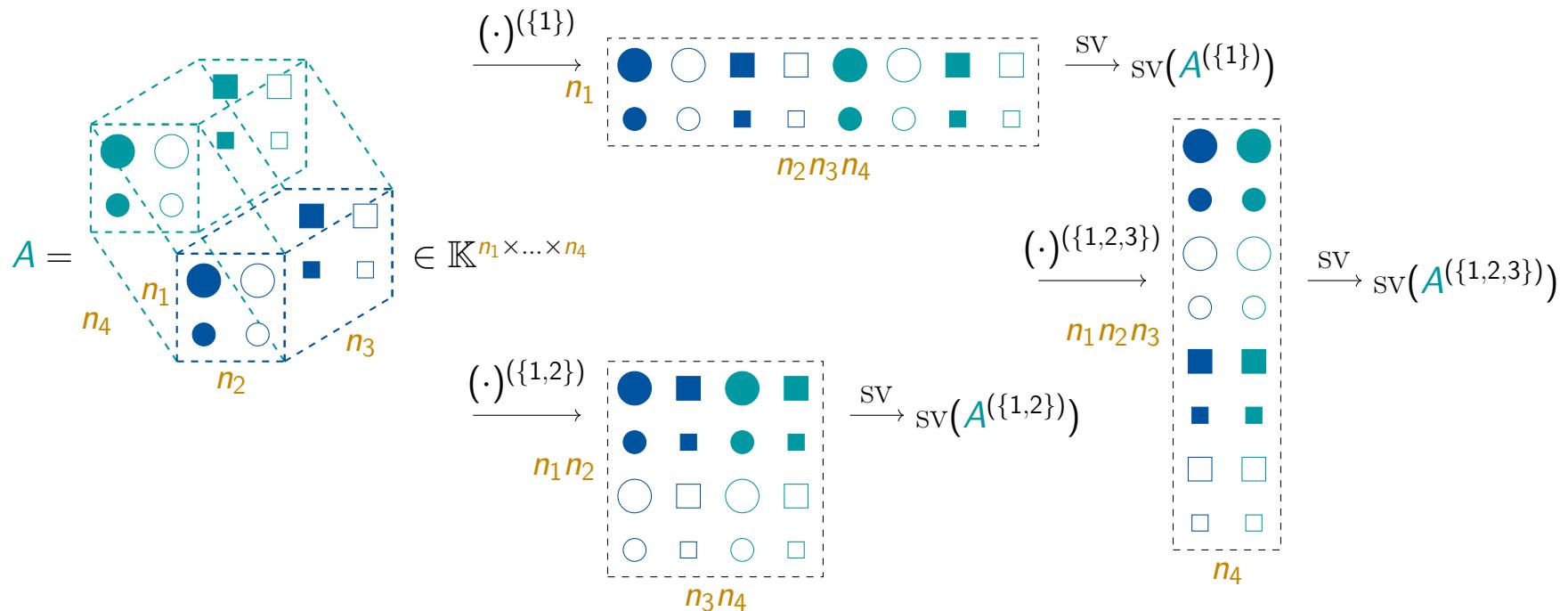
Feasibility and Compatibility — Reshaping



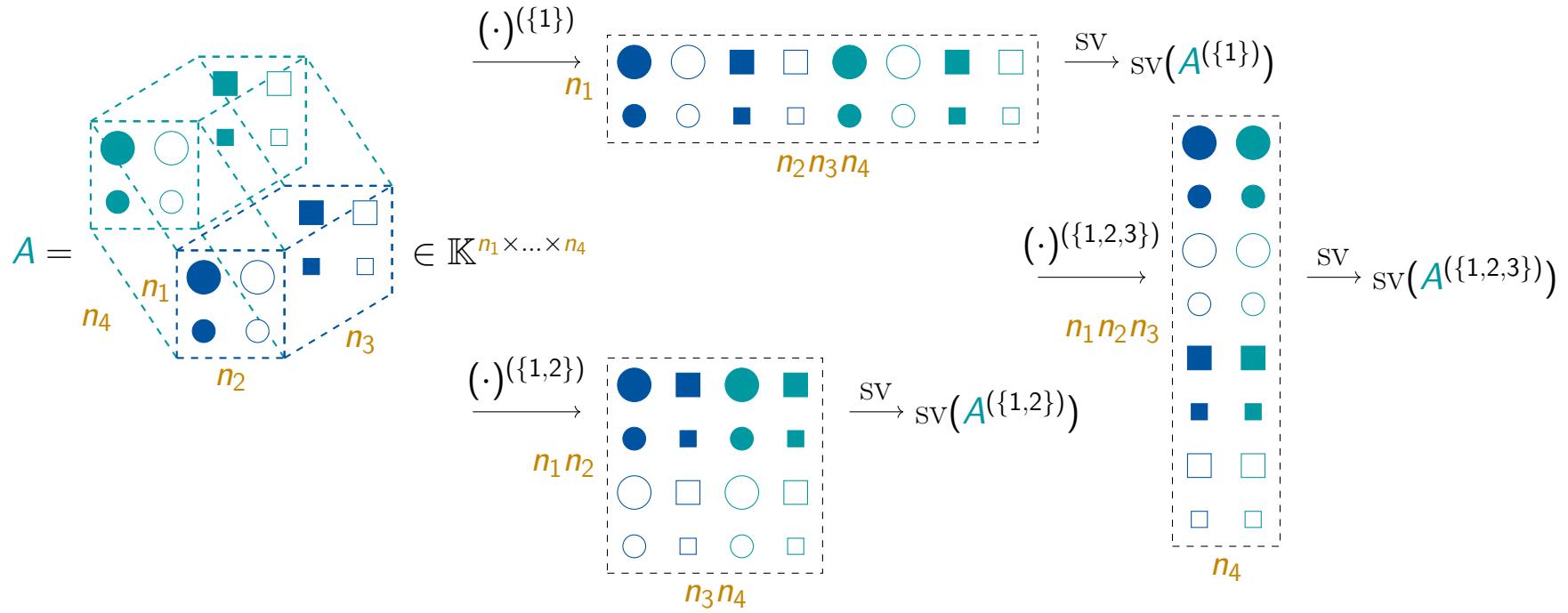
Feasibility and Compatibility — Reshaping



Feasibility and Compatibility — Reshaping



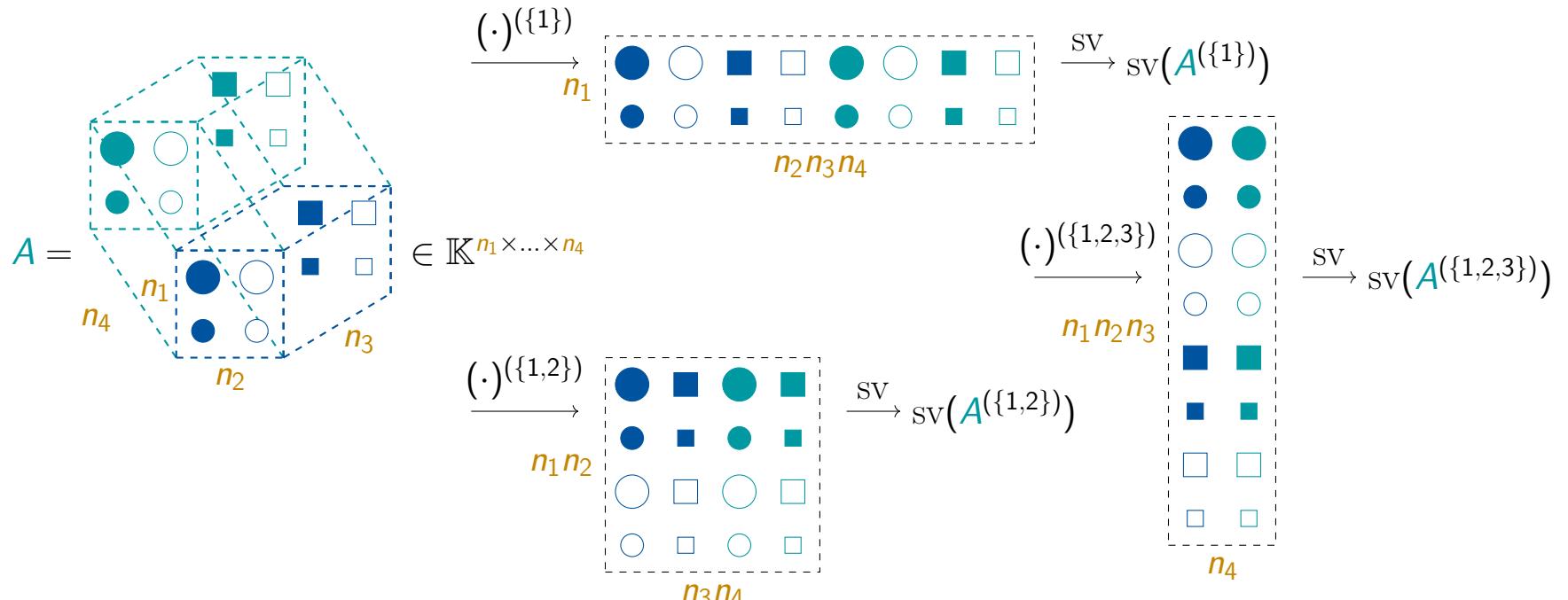
Feasibility and Compatibility — Reshaping



$$\mathbb{K}^{n_1 \times n_2 \times n_3 \times n_4} \cong \mathbb{K}^{n_1 \times (n_2 n_3 n_4)} \cong \mathbb{K}^{(n_1 n_2) \times (n_3 n_4)} \cong \mathbb{K}^{(n_1 n_2 n_3) \times n_4}$$

$$\cong \mathbb{K}^{(n_1 n_3) \times (n_2 n_4)} \cong \dots$$

Feasibility and Compatibility — Reshaping



\mathcal{K} —family of singular values:

$$\{\text{sv}(\mathcal{A}^{(J)})\}_{J \in \mathcal{K}} \quad \text{for} \quad J \subset \{1, \dots, d-1\}$$

in the picture:

$$\mathcal{K} = \{ \{1\}, \{1, 2\}, \{1, 2, 3\} \} \quad (\text{tensor train for } d=4)$$

Feasibility and Compatibility — Tensor Feasibility Problem (TFP)

Tensor feasibility problem (TFP):

Definition (Feasibility)

$\{\sigma^{(J)}\}_{J \in \mathcal{K}}$ is feasible for $n \in \mathbb{N}^d \Leftrightarrow \exists A \in \mathbb{K}^{n_1 \times \dots \times n_d}$, $\text{sv}(A^{(J)}) = \sigma^{(J)}$ for all $J \in \mathcal{K}$

Feasibility and Compatibility — Tensor Feasibility Problem (TFP)

Tensor feasibility problem (TFP):

Definition (Feasibility)

$\{\sigma^{(J)}\}_{J \in \mathcal{K}}$ is feasible for $n \in \mathbb{N}^d \Leftrightarrow \exists A \in \mathbb{K}^{n_1 \times \dots \times n_d}$, $\text{sv}(A^{(J)}) = \sigma^{(J)}$ for all $J \in \mathcal{K}$

Trace condition for feasibility: $\|\sigma^{(I)}\| = \|\sigma^{(J)}\|$ for all $I, J \in \mathcal{K}$

proof: $\text{sv}(A^{(J)}) = \sigma^{(J)} \Rightarrow \sum_{i \in \mathbb{N}} (\sigma_i^{(J)})^2 = \|A\|_F$

Feasibility and Compatibility — Tensor Feasibility Problem (TFP)

Tensor feasibility problem (TFP):

Definition (Feasibility)

$\{\sigma^{(J)}\}_{J \in \mathcal{K}}$ is feasible for $n \in \mathbb{N}^d \Leftrightarrow \exists A \in \mathbb{K}^{n_1 \times \dots \times n_d}$, $\text{sv}(A^{(J)}) = \sigma^{(J)}$ for all $J \in \mathcal{K}$

Trace condition for feasibility: $\|\sigma^{(I)}\| = \|\sigma^{(J)}\|$ for all $I, J \in \mathcal{K}$

proof: $\text{sv}(A^{(J)}) = \sigma^{(J)} \Rightarrow \sum_{i \in \mathbb{N}} (\sigma_i^{(J)})^2 = \|A\|_F$

Example: $\mathcal{K} = \{ \{1\}, \{1, 2\}, \{1, 2, 3\} \}$

- ▶ $(\sigma^{(\{1\})})^2 = (9, 4.5)$
- ▶ $(\sigma^{(\{1, 2\})})^2 = (10, 2, 1, 0.25, 0.25)$
- ▶ $(\sigma^{(\{1, 2, 3\})})^2 = (4, 3, 2.5, 2, 2)$

Question: σ feasible for some $n \in \mathbb{N}^4$? / can we construct such a tensor A ?

Why ask: complexity analysis / similarity of decay / better intuition

Feasibility and Compatibility — Feasibility for the Tucker Format

Results for the Tucker-format, $\mathcal{K} = \{\{1\}, \{2\}, \dots, \{d\}\}$:

- ▶ Introduction and first contributions to the Tucker TFP:
 - ▷ [HaKrUs17] : *W. Hackbusch, D. Kressner, A. Uschmajew, Perturbation of higher-order singular values, SIAGA (2017)*
 - ▷ [HaUs17] : *W. Hackbusch, A. Uschmajew, In the interconnection between the higher-order singular values of real tensors, Numerische Mathematik (2017)*

Feasibility and Compatibility — Feasibility for the Tucker Format

Results for the Tucker-format, $\mathcal{K} = \{\{1\}, \{2\}, \dots, \{d\}\}$:

- ▶ Introduction and first contributions to the Tucker TFP:
 - ▷ [HaKrUs17] : W. Hackbusch, D. Kressner, A. Uschmajew, *Perturbation of higher-order singular values, SIAGA* (2017)
 - ▷ [HaUs17] : W. Hackbusch, A. Uschmajew, *In the interconnection between the higher-order singular values of real tensors, Numerische Mathematik* (2017)
- ▶ Solution for largest singular values:
 - ▷ [DoStLa17] : I. Domanov, A. Stegeman, L. De Lathauwer, *On the largest multilinear singular values of higher-order tensors, SIMAX* (2017)
- ▶ Independent solution for binary tensors:
 - ▷ [Se18] : A. Seigal, *Gram determinants of real binary tensors, Linear Algebra and its Applications* (2018)

Feasibility and Compatibility — Feasibility for the Tucker Format

(a version of the) Quantum marginal problem (QMP) [Kl06][DaHa05][Sc14]:

Definition (Compatibility)

$\{\lambda^{(J)}\}_{J \in \mathcal{K}}$ is compatible for $n \in \mathbb{N}^{d-1} \Leftrightarrow$ there exists a hermitian, positive semi-definite matrix $\rho \in \mathbb{C}^{(n_1 \dots n_{d-1}) \times (n_1 \dots n_{d-1})}$ such that

$$\text{ev}(\text{trace}_{\{1, \dots, d-1\} \setminus J}(\rho)) = \lambda^{(J)}$$

for all $J \in \mathcal{K}$.

Pure quantum marginal problem (pure QMP) adds $\text{rank}(\rho) = 1$.
The partial traces are called *marginals* of the *density operator* ρ .

The QMP is much older than the TFP !

[Kl06] : A. A. Klyachko, Quantum marginal problem and N-representability, *Journal of Physics* (2006)

[DaHa05] : S. Daftuar and P. Hayden, Quantum state transformations and the Schubert calculus, *Annals of Physics*, (2005)

[Sc14] : C. Schilling, Quantum marginal problem and its physical relevance, *PhD thesis* (2014)

Feasibility and Compatibility — Equivalence of TFP and QMP

Tensor feasibility (TFP) and quantum marginal problem (QMP) are equivalent:

Theorem (Equivalence)

$$\{(\sigma^{(J)})^2\}_{J \in \mathcal{K}} \text{ is compatible} \Leftrightarrow \{\sigma^{(J)}\}_{J \in \mathcal{K}} \text{ is feasible (for } n_d = \text{rank}(\rho))$$

proof: $\rho = A^{(I)} A^{(I)H}$, $I = \{1, \dots, d-1\}$

Feasibility and Compatibility — Equivalence of TFP and QMP

Tensor feasibility (TFP) and quantum marginal problem (QMP) are equivalent:

Theorem (Equivalence)

$$\{(\sigma^{(J)})^2\}_{J \in \mathcal{K}} \text{ is compatible} \Leftrightarrow \{\sigma^{(J)}\}_{J \in \mathcal{K}} \text{ is feasible (for } n_d = \text{rank}(\rho))$$

proof: $\rho = A^{(I)} A^{(I)H}$, $I = \{1, \dots, d-1\}$

Solutions for three dimensions:

- ▶ $\mathcal{K} = \{\{1\}, \{2\}, \{3\}\}$: pure QMP [Kl06] (\rightsquigarrow Tucker-feasibility for $d=3$)
- ▶ $\mathcal{K} = \{\{1\}, \{1, 2\}\}$: QMP [DaHa05] (\rightsquigarrow Tensor Train-feasibility for $d=3$)

These sets of compatible values form convex, polyhedral cones.

QMP results allow to calculate H -description for each fixed n_1, n_2 ,
but do not (seem to) yield universal inequalities for arbitrary n .

Tensor Tree Decompositions / SVDs —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

TT-Feasibility

Tensor Tree Decompositions / SVDs — Matrix Decompositions

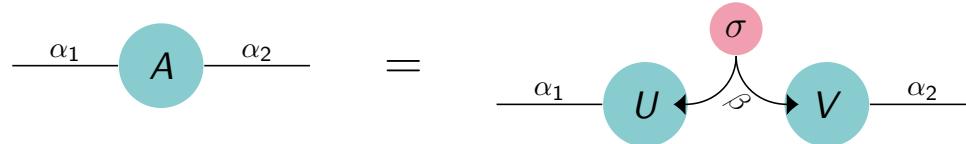
$d = 2$: we can visualize the low-rank decomposition of $A \in \mathbb{K}^{n_1 \times n_2}$, $r = \text{rank}(A)$,

$$A_{\alpha_1, \alpha_2} = \sum_{\beta=1}^r X_{\alpha_1, \beta} Y_{\alpha_2, \beta}$$

as *tensor network*:



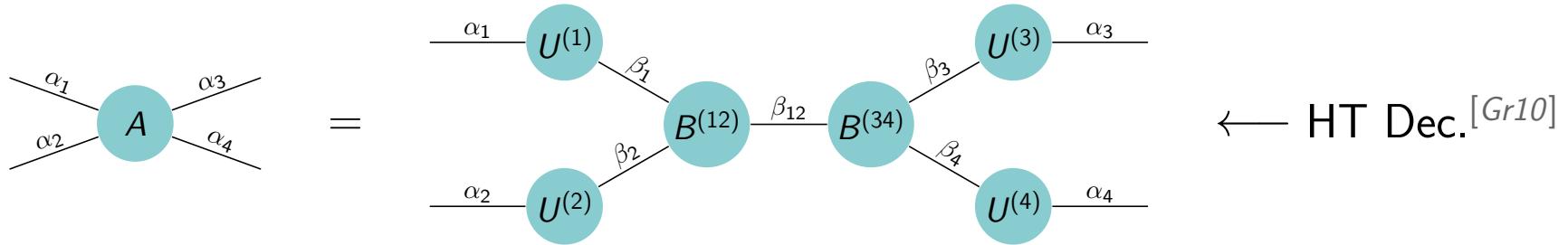
$$A_{\alpha_1, \alpha_2} = \sum_{\beta=1}^r U_{\alpha_1, \beta} \sigma_\beta V_{\alpha_2, \beta} \quad (\text{SVD})$$



- ▶ U and V are column-unitary

Tensor Tree Decompositions / SVDs — Tensor Decompositions

$r_J := \text{rank}(A^{(J)}) = \deg(\sigma^{(J)}), \quad \beta_J = 1, \dots, r_J$:

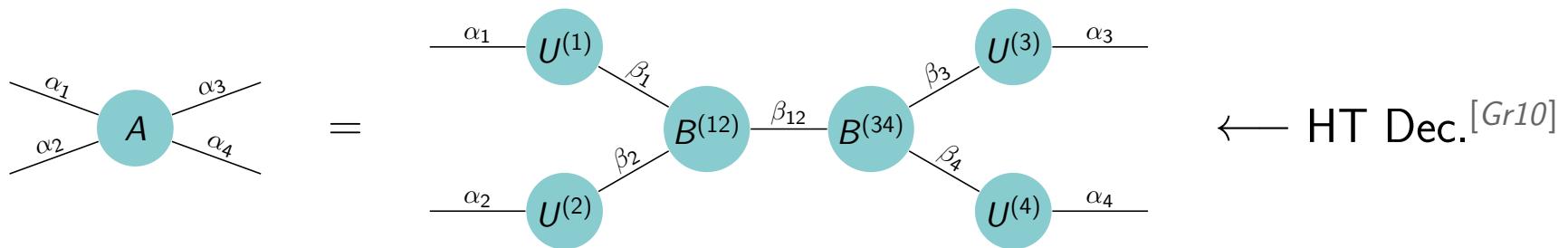


[Gr10] : L. Grasedyck, *Hierarchical SVD of Tensors*, SIAM J. Matrix Anal. & Appl (2010)

Tensor Tree Decompositions / SVDs — Tensor Decompositions

$$r_J := \text{rank}(A^{(J)}) = \deg(\sigma^{(J)}), \quad \beta_J = 1, \dots, r_J:$$

$$A_{\alpha_1, \dots, \alpha_4} = \sum_{\beta_1=1}^{r_{\{1\}}} \cdots \sum_{\beta_4=1}^{r_{\{4\}}} \sum_{\beta_{12}=1}^{r_{\{1,2\}}} U_{\alpha_1, \beta_1}^{(\{1\})} \cdots U_{\alpha_4, \beta_4}^{(\{4\})} \cdot B_{\beta_1, \beta_2, \beta_{12}}^{(12)} \cdot B_{\beta_3, \beta_4, \beta_{12}}^{(34)}$$

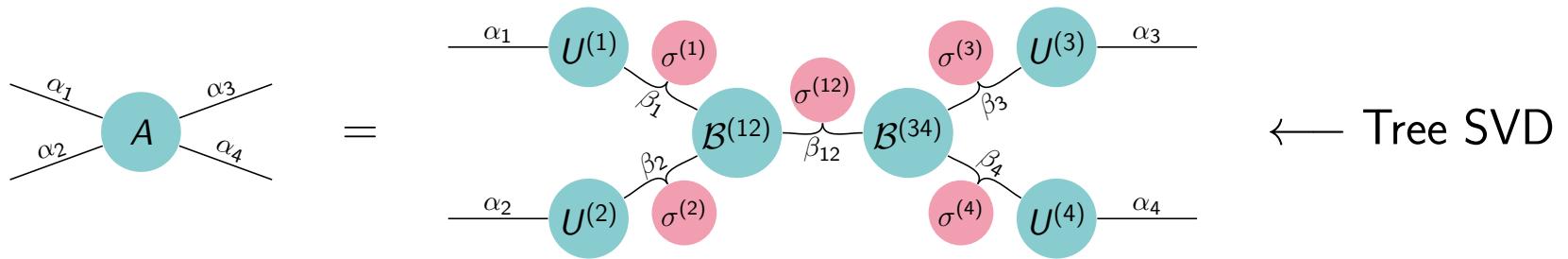


[Gr10] : L. Grasedyck, *Hierarchical SVD of Tensors*, SIAM J. Matrix Anal. & Appl (2010)

Tensor Tree Decompositions / SVDs — Tensor Decompositions

$$r_J := \text{rank}(A^{(J)}) = \deg(\sigma^{(J)}), \quad \beta_J = 1, \dots, r_J:$$

$$A_{\alpha_1, \dots, \alpha_4} = \sum_{\beta_1=1}^{r_{\{1\}}} \cdots \sum_{\beta_4=1}^{r_{\{4\}}} \sum_{\beta_{12}=1}^{r_{\{1,2\}}} U_{\alpha_1, \beta_1}^{(\{1\})} \cdots U_{\alpha_4, \beta_4}^{(\{4\})} \cdot \mathcal{B}_{\beta_1, \beta_2, \beta_{12}}^{(12)} \cdot \mathcal{B}_{\beta_3, \beta_4, \beta_{12}}^{(34)} \\ \cdot \sigma_{\beta_1}^{(\{1\})} \cdots \sigma_{\beta_4}^{(\{4\})} \sigma_{\beta_{12}}^{(\{1,2\})}$$

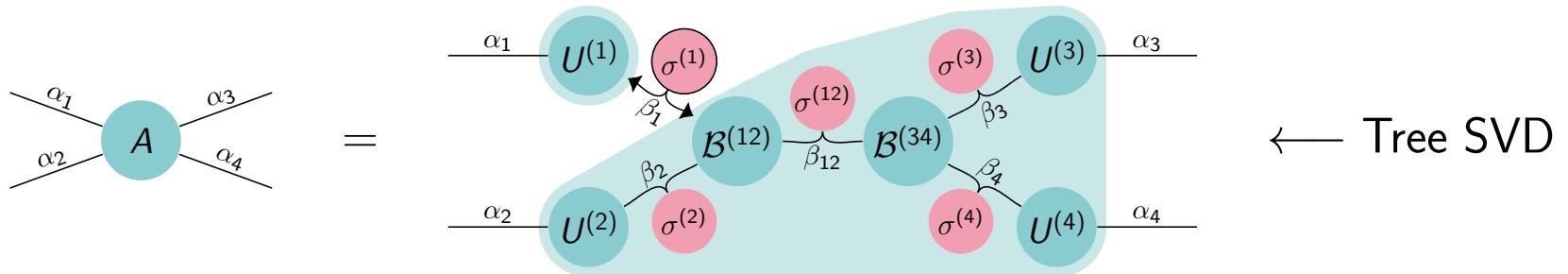


[Gr10] : L. Grasedyck, Hierarchical SVD of Tensors, SIAM J. Matrix Anal. & Appl (2010)

Tensor Tree Decompositions / SVDs — Tensor Decompositions

$$r_J := \text{rank}(A^{(J)}) = \deg(\sigma^{(J)}), \quad \beta_J = 1, \dots, r_J:$$

$$A_{\alpha_1, \dots, \alpha_4} = \sum_{\beta_1=1}^{r_{\{1\}}} U_{\alpha_1, \beta_1}^{(\{1\})} \sigma_{\beta_1}^{(\{1\})} \cdot \underbrace{\left(\sum_{\beta_2=1}^{r_{\{2\}}} \cdots \sum_{\beta_{12}=1}^{r_{\{1,2\}}} U_{\alpha_2, \beta_2}^{(\{2\})} \cdots B_{\beta_3, \beta_4, \beta_{12}}^{(34)} \cdots \sigma_{\beta_{12}}^{(\{1,2\})} \right)}_{V_{(\alpha_2, \alpha_3, \alpha_4), \beta_1}^{(\{1\})}}$$



- $(U^J, \sigma^{(J)}, V^{(J)}) = \text{SVD}(A^{(J)}), J \in \mathcal{K}$
- $\mathcal{K} = \{ \{1, 2\}, \{1\}, \{2\}, \{3\}, \{4\} \} \rightarrow \text{binary hierarchical Tucker format}$

The tree SVD puts gauge conditions on inner nodes:

Theorem (Decoupling)

Let \mathcal{K} fulfill the hierarchy condition ^[Gr10] (it corresponds to a tree graph)

$$J \cap \tilde{J} \in \{\emptyset, J, \tilde{J}\}, \quad \text{for all } J, \tilde{J} \in \mathcal{K}.$$

Then the conditions for feasibility can be decoupled into two and three dimensional subproblems.

The tree SVD puts gauge conditions on inner nodes:

Theorem (Decoupling)

Let \mathcal{K} fulfill the hierarchy condition ^[Gr10] (it corresponds to a tree graph)

$$J \cap \tilde{J} \in \{\emptyset, J, \tilde{J}\}, \quad \text{for all } J, \tilde{J} \in \mathcal{K}.$$

Then the conditions for feasibility can be decoupled into two and three dimensional subproblems. With ^{[KI06][DaHa05]}, we thereby know that squared feasible values for such \mathcal{K} form closed, convex, polyhedral cones.

The tree SVD puts gauge conditions on inner nodes:

Theorem (Decoupling)

Let \mathcal{K} fulfill the hierarchy condition ^[Gr10] (it corresponds to a tree graph)

$$J \cap \tilde{J} \in \{\emptyset, J, \tilde{J}\}, \quad \text{for all } J, \tilde{J} \in \mathcal{K}.$$

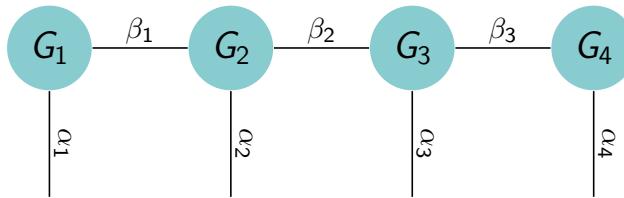
Then the conditions for feasibility can be decoupled into two and three dimensional subproblems. With ^{[KI06][DaHa05]}, we thereby know that squared feasible values for such \mathcal{K} form closed, convex, polyhedral cones.

Example: $\mathcal{K} = \{\{1, 2\}, \{1\}, \{2\}, \{3\}, \{4\}\}$

$\{\sigma^{(J)}\}_{J \in \mathcal{K}}$ feasible for $n \in \mathbb{N}^4 \Leftrightarrow \{\sigma^{(\{1\})}, \sigma^{(\{2\})}, \sigma^{(\{1,2\})}\}$ feasible for $(n_1, n_2, r_{\{1,2\}})$
and $\{\sigma^{(\{3\})}, \sigma^{(\{4\})}, \sigma^{(\{1,2\})}\}$ feasible for $(n_3, n_4, r_{\{1,2\}})$

Tensor Tree Decompositions / SVDs — TT and Tucker

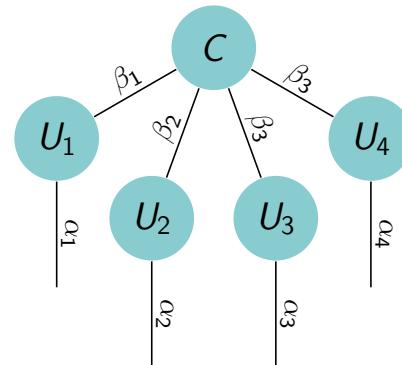
Tensor Train (TT)



$$\mathcal{K} = \{ \{1\}, \{1, 2\}, \{1, 2, 3\} \},$$

TT tree dec. \equiv TTSVD^[Os11] / TT tree SVD \equiv canonical MPS^[Vi03]

Tucker



$$\mathcal{K} = \{ \{1\}, \{2\}, \{3\}, \{4\} \}$$

Tucker tree dec. \equiv HOSVD / Tucker tree SVD \rightarrow all-orthogonality^[LaMoVa00]

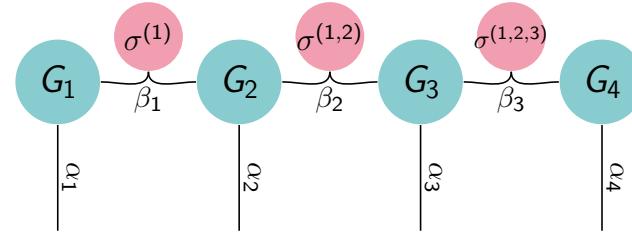
[Os11] : I. Oseledets, *Tensor-Train Decomposition*, SIAM J. Sci. Comput. (2011)

[Vi03] : G. Vidal, *Efficient Classical Simulation of Slightly Entangled Quantum Computations*, Physical Review Letters (2003)

[LaMoVa00] : L. Lathauwer, B. Moor, J. Vandewalle, *A multilinear SVD*, Siam J. Matrix Anal. Appl. (2000)

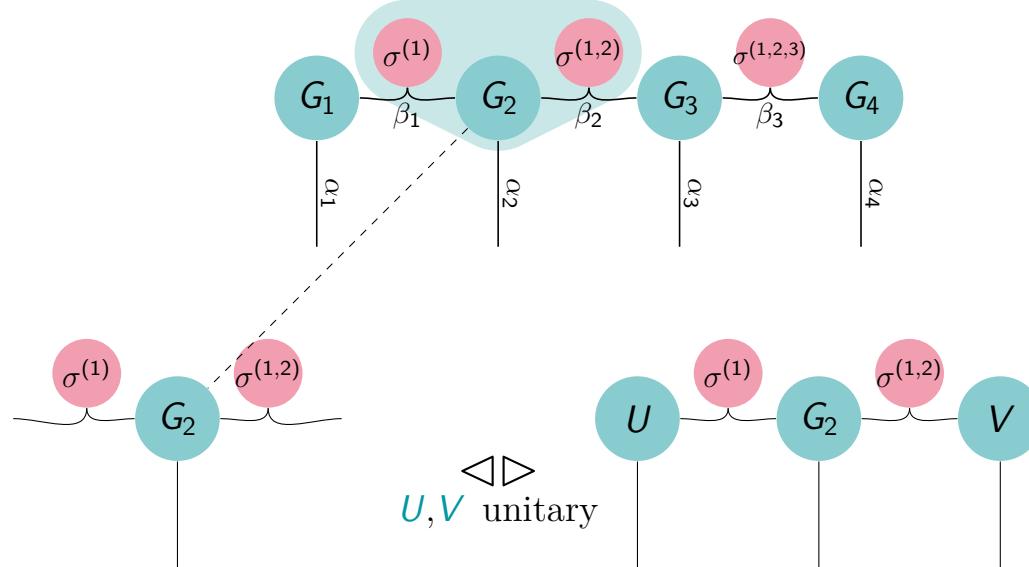
Tensor Tree Decompositions / SVDs — Decoupling for TT

Tree SVD for the tensor train format:



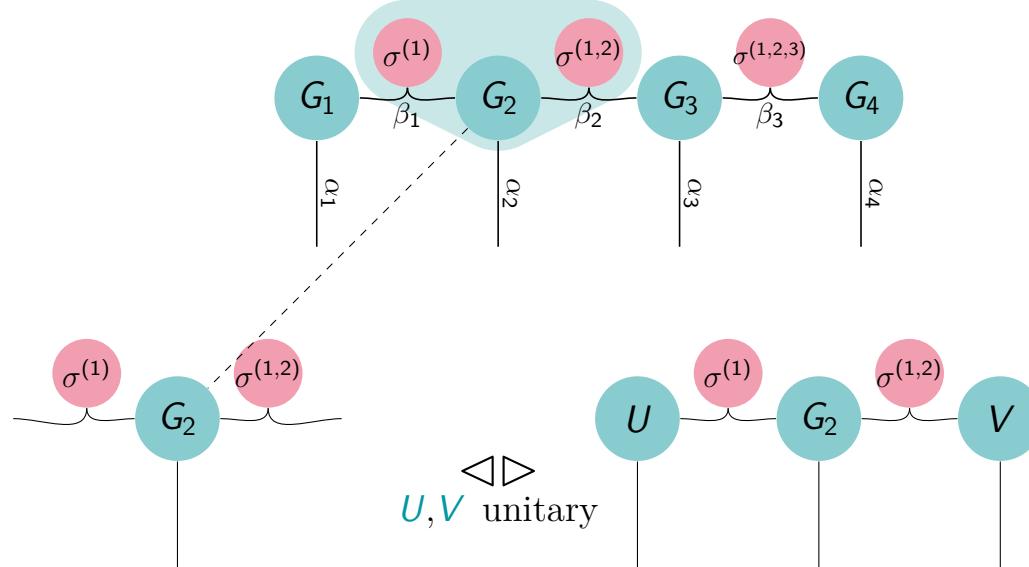
Tensor Tree Decompositions / SVDs — Decoupling for TT

Tree SVD for the tensor train format:



Tensor Tree Decompositions / SVDs — Decoupling for TT

Tree SVD for the tensor train format:



$G_2 = G_2(\alpha_2)_{\beta_1, \beta_2}$ fulfills simple constraints:

$$\sum_{\alpha_2} G_2(\alpha_2)^H \operatorname{diag}(\sigma^{(\{1\})})^2 G_2(\alpha_2) = I, \quad \sum_{\alpha_2} G_2(\alpha_2) \operatorname{diag}(\sigma^{(\{1,2\})})^2 G_2(\alpha_2)^H = I$$

Corollary (TT-feasibility)

Let $\mathcal{K} = \{\{1\}, \{1, 2\}, \dots, \{1, \dots, d-1\}\}$. Then $\{\sigma^{(J)}\}_{J \in \mathcal{K}}$ is feasible

if and only if

each neighboring pair $(\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$ is feasible for $(r_{\{1, \dots, \mu-1\}}, n_\mu, r_{\{1, \dots, \mu\}})$ and $\mathcal{K} = \{\{1\}, \{1, 2\}\}, \mu = 2, \dots, d-1$ (as well as $r_{\{1\}} \leq n_1$ and $r_{\{1, \dots, d-1\}} \leq n_d$).

Corollary (TT-feasibility)

Let $\mathcal{K} = \{\{1\}, \{1, 2\}, \dots, \{1, \dots, d-1\}\}$. Then $\{\sigma^{(J)}\}_{J \in \mathcal{K}}$ is feasible

if and only if

each neighboring pair $(\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$ is feasible for $(r_{\{1, \dots, \mu-1\}}, n_\mu, r_{\{1, \dots, \mu\}})$ and $\mathcal{K} = \{\{1\}, \{1, 2\}\}, \mu = 2, \dots, d-1$ (as well as $r_{\{1\}} \leq n_1$ and $r_{\{1, \dots, d-1\}} \leq n_d$).

For pairs, we just write *feasible for $m (= n_\mu) \in \mathbb{N}$* . The decoupling is constructive.

We will turn the feasibility of pairs into an interlinked eigenvalue relation.

Later yields: conditions for TT-feasibility are the same for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Sums of Hermitian Matrices and Honeycombs —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

TT-Feasibility

Sums of Hermitian Matrices and Honeycombs — Weyl's Problems

- Weyls problem [We1912]: Given $(a, b, c) \in \mathbb{R}^{3k}$, do there exist hermitian matrices $A, B, C \in \mathbb{R}^{k \times k}$ with $A + B = C$ and $a = \text{ev}(A), \dots, c = \text{ev}(C)$:


e.v.

$$A + B = C \in \mathbb{C}^{k \times k} \text{ (hermitian)}$$

a	b	c
---	---	---

[We1912] : H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (...), Math. Annalen (1912)
[Ho62] : A. Horn, Eigenvalues of sums of Hermitian matrices, Pacific J. Math. (1962)
[KnTa01] : A. Knutson, T. Tao, Honeycombs and sums of Hermitian matrices, Notices. Amer. Math. Soc. (2001)

Sums of Hermitian Matrices and Honeycombs — Weyl's Problems

- Weyls problem [We1912]: Given $(a, b, c) \in \mathbb{R}^{3k}$, do there exist hermitian matrices $A, B, C \in \mathbb{R}^{k \times k}$ with $A + B = C$ and $a = \text{ev}(A), \dots, c = \text{ev}(C)$:

$$\begin{array}{c} \text{e.v.} \\ | \\ \text{e.v.} \end{array} \quad \begin{array}{ccc} A & + & B \\ | & & | \\ a & & b \end{array} = \begin{array}{c} C \\ | \\ c \end{array} \quad \in \mathbb{C}^{k \times k} \text{ (hermitian)}$$

- Horn Conjecture [Ho62] (verified [KnTa01]): the set $F \subset \mathbb{R}^{3k}$ of such (a, b, c) is a polyhedral cone (governed by simple, but recursively defined inequalities).

[We1912] : H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (...)*, Math. Annalen (1912)
[Ho62] : A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math. (1962)
[KnTa01] : A. Knutson, T. Tao, *Honeycombs and sums of Hermitian matrices*, Notices. Amer. Math. Soc. (2001)

Sums of Hermitian Matrices and Honeycombs — Weyl's Problems

- Weyls problem [We1912]: Given $(a, b, c) \in \mathbb{R}^{3k}$, do there exist hermitian matrices $A, B, C \in \mathbb{R}^{k \times k}$ with $A + B = C$ and $a = \text{ev}(A)$, \dots , $c = \text{ev}(C)$:

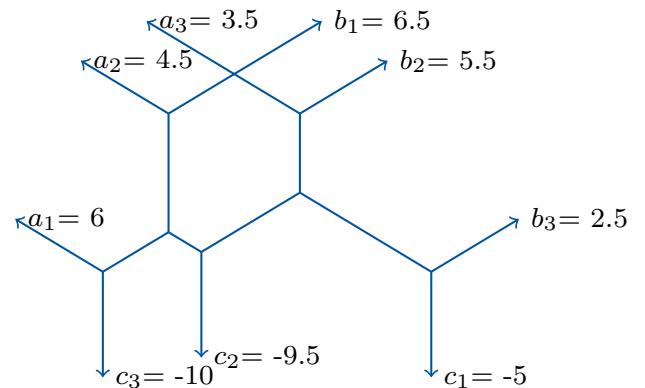
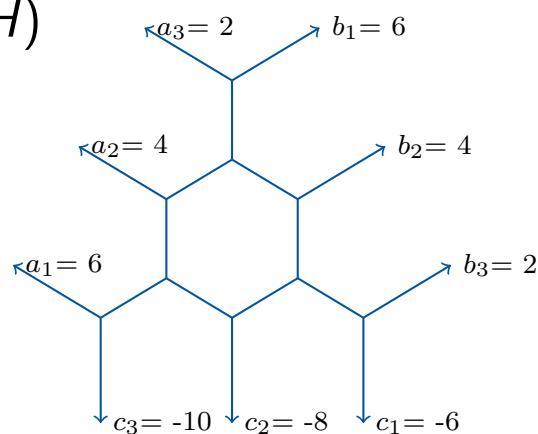
e.v.

$$A + B = C \in \mathbb{C}^{k \times k} \text{ (hermitian)}$$

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$

- Horn Conjecture [Ho62] (verified [KnTa01]): the set $F \subset \mathbb{R}^{3k}$ of such (a, b, c) is a polyhedral cone (governed by simple, but recursively defined inequalities).
- $(a, b, c) \in F \Leftrightarrow$ there is a *honeycomb* H with boundary $(a, b, -c)$ [KnTa01].

$$F = \text{Proj}_{\text{boundary}}(H)$$



Sums of Hermitian Matrices and Honeycombs — Hives

- ▶ The triplets (a, b, c) are the same for
(complex, hermitian) \leftrightarrow (real, symmetric) matrices [Fu00]

[Fu00] : W. Fulton, Eigenvalues of majorized Hermitian matrices and LittlewoodRichardson coefficients, *Linear Algebra and its Applications* (2000)

Sums of Hermitian Matrices and Honeycombs — Hives

- ▶ The triplets (a, b, c) are the same for
(complex, hermitian) \leftrightarrow (real, symmetric) matrices [Fu00]

- ▶ Chains of matrices [KnTa01]:

$$\begin{array}{c} \text{e.v.} \\ | \\ a^{(1)} \end{array} \quad A^{(1)} + A^{(2)} + \cdots + A^{(\textcolor{brown}{m})} = \quad C \quad \in \mathbb{C}^{k \times k} \text{ (hermitian)}$$
$$\begin{array}{c} | \\ a^{(2)} \\ | \\ \vdots \\ | \\ a^{(\textcolor{brown}{m})} \\ | \\ c \end{array}$$

[Fu00] : W. Fulton, Eigenvalues of majorized Hermitian matrices and LittlewoodRichardson coefficients, Linear Algebra and its Applications (2000)

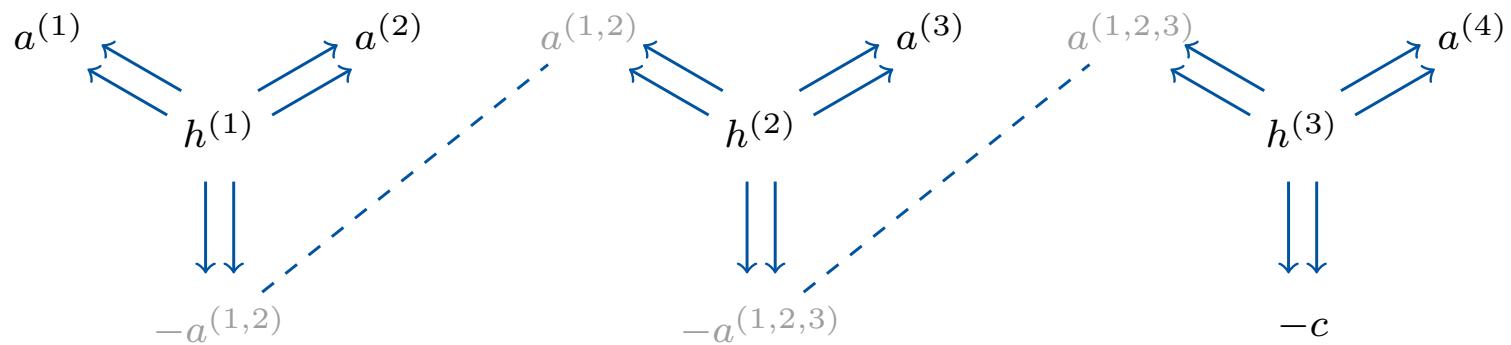
Sums of Hermitian Matrices and Honeycombs — Hives

- ▶ The triplets (a, b, c) are the same for
(complex, hermitian) \leftrightarrow (real, symmetric) matrices [Fu00]

- ▶ Chains of matrices [KnTa01]:

$$\begin{array}{c} \text{e.v.} \\ | \\ a^{(1)} \end{array} \quad A^{(1)} + A^{(2)} + \cdots + A^{(\textcolor{brown}{m})} = \quad C \quad \in \mathbb{C}^{k \times k} \text{ (hermitian)}$$
$$| \qquad \qquad \qquad | \qquad \qquad \qquad | \qquad \qquad \qquad | \qquad \qquad \qquad |$$
$$a^{(2)} \qquad \qquad \qquad a^{(\textcolor{brown}{m})} \qquad \qquad \qquad c$$

- ▶ Form chains of honeycombs $H = (h^{(1)}, h^{(2)}, \dots, h^{(\textcolor{brown}{m})})$ (hives) [KnTa01]:



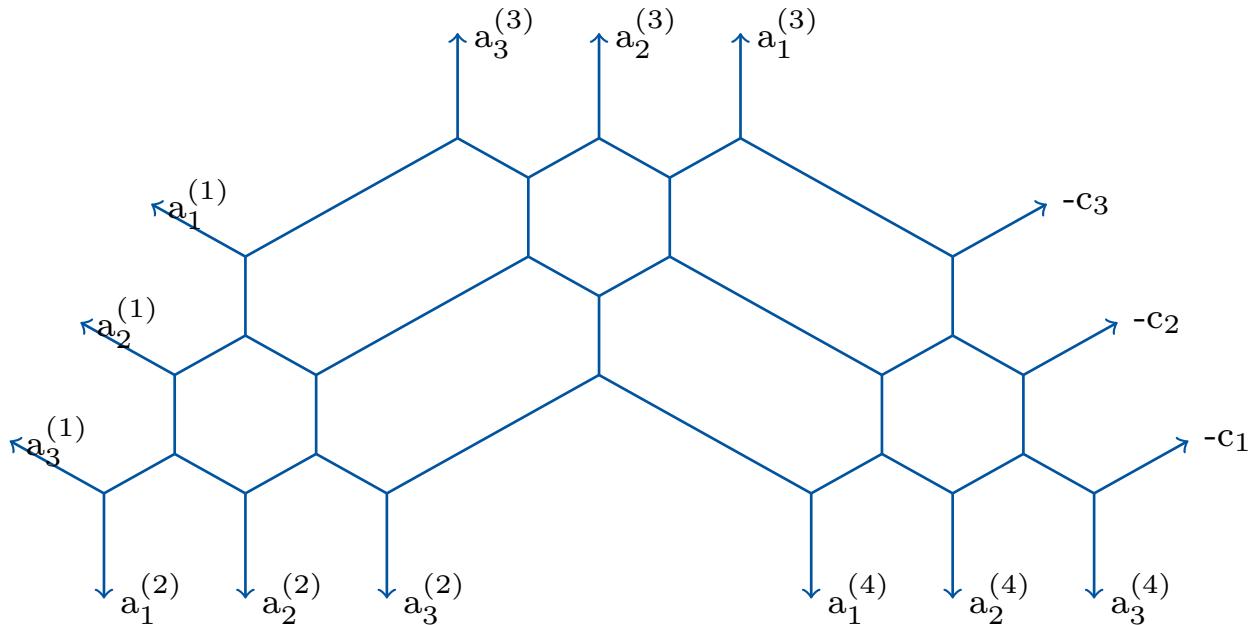
[Fu00] : W. Fulton, Eigenvalues of majorized Hermitian matrices and LittlewoodRichardson coefficients, Linear Algebra and its Applications (2000)

Sums of Hermitian Matrices and Honeycombs — Hives

- ▶ Chains of matrices [KnTa01]:

$$\begin{array}{c} \text{○} \\ | \\ \text{e.v.} \end{array} \quad A^{(1)} + A^{(2)} + \cdots + A^{(m)} = C \in \mathbb{C}^{k \times k} \text{ (hermitian)}$$
$$a^{(1)} \mid a^{(2)} \mid \cdots \mid a^{(m)} \mid c$$

- ▶ Form chains of honeycombs $H = (h^{(1)}, h^{(2)}, \dots, h^{(m)})$ (*hives*) [KnTa01]:



Feasibility of Pairs —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

TT-Feasibility

Feasibility of Pairs — Connection to Hives

Theorem (Conversion to eigenvalue problem (constructive))

(γ, θ) feasible for m if and only if for some $k \geq \deg(\gamma), \deg(\theta)$ there exist hermitian, positive semi-definite matrices $A^{(i)}, B^{(i)}, C, D$ in following relation:

$$\begin{array}{c} \text{e.v.} \\ \lambda^{(1)} \quad \lambda^{(2)} \quad \lambda^{(m)} \end{array} \left| \begin{array}{l} A^{(1)} + A^{(2)} + \cdots + A^{(m)} = C \in \mathbb{C}^{k \times k} \text{ (hermitian)} \\ \geq 0 \\ B^{(1)} + B^{(2)} + \cdots + B^{(m)} = D \in \mathbb{C}^{k \times k} \text{ (hermitian)} \end{array} \right| \begin{array}{c} \gamma^2 \\ \theta^2 \end{array}$$

Feasibility of Pairs — Connection to Hives

Theorem (Conversion to eigenvalue problem (constructive))

(γ, θ) feasible for m if and only if for some $k \geq \deg(\gamma), \deg(\theta)$ there exist hermitian, positive semi-definite matrices $A^{(i)}, B^{(i)}, C, D$ in following relation:

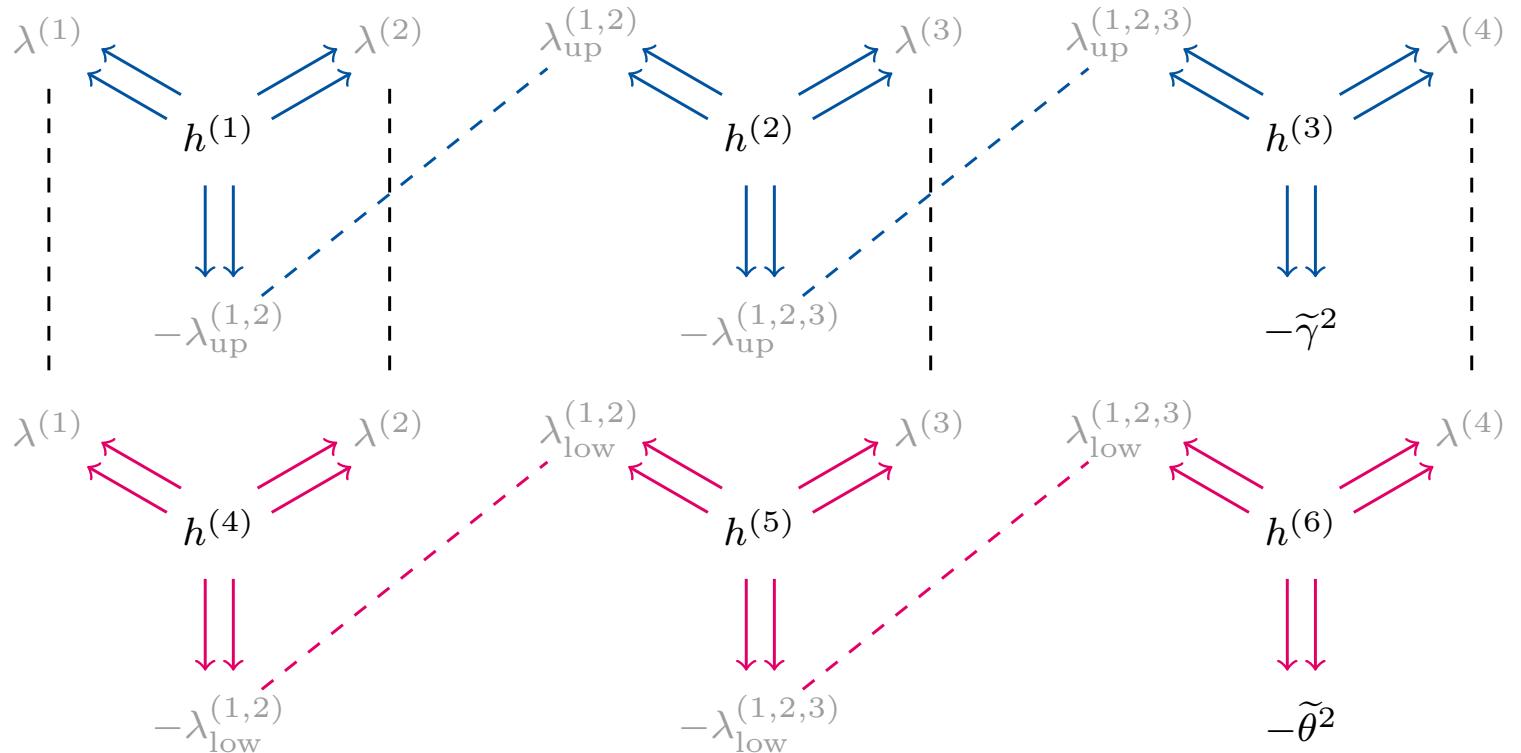
$$\begin{array}{c} \text{e.v.} \\ \lambda^{(1)} \quad \lambda^{(2)} \quad \lambda^{(m)} \end{array} \left| \begin{array}{l} A^{(1)} + A^{(2)} + \cdots + A^{(m)} = C \in \mathbb{C}^{k \times k} \text{ (hermitian)} \\ \geq 0 \\ B^{(1)} + B^{(2)} + \cdots + B^{(m)} = D \in \mathbb{C}^{k \times k} \text{ (hermitian)} \end{array} \right| \begin{array}{c} \gamma^2 \\ \theta^2 \end{array}$$

Lemma (Restriction to diagonal matrices)

If $\deg(\gamma), \deg(\theta) \leq m$, then we can always construct such matrices, as well as a corresponding node.

Feasibility of Pairs — Connection to Hives

- Form coupled chains of honeycombs $H = (h^{(1)}, h^{(2)}, \dots, h^{(2m)})$ (*hives*):



- Conditions are given by a system of linear inequalities (also proving the cone property).

Feasibility of Pairs — Necessary Inequalities

We can derive classes of necessary inequalities, including:

Lemma

If (γ, θ) is feasible for m , then for every k

$$\sum_{i=1}^k \gamma_i^2 \leq \sum_{i=1}^{mk} \theta_i^2 \quad \text{as in [DaHa05]} \quad (1)$$

and

$$\gamma_{k+m+1}^2 \leq \sum_{i=k+1}^{k+m} \theta_i^2.$$

Feasibility of Pairs — Necessary Inequalities

We can derive classes of necessary inequalities, including:

Lemma

If (γ, θ) is feasible for m , then for every k

$$\sum_{i=1}^k \gamma_i^2 \leq \sum_{i=1}^{mk} \theta_i^2 \quad \text{as in [DaHa05]} \quad (1)$$

and

$$\gamma_{k+m+1}^2 \leq \sum_{i=k+1}^{k+m} \theta_i^2.$$

A special case has been conjectured in QMP literature^[DaHa05], and then proven by:

Theorem ([LiPoWa14])

If $\deg(\gamma) \leq m$ and (1) holds, then the pair (γ, θ) is feasible for m .

[LiPoWa14] : C.-K. Li, Y.-T. Poon, and X. Wang, Ranks and ev. of states with prescribed reduced states, Electronic J. of Linear Algebra (2014)

Feasibility of Pairs — Necessary Inequalities

We can derive classes of necessary inequalities, including:

Lemma

If (γ, θ) is feasible for m , then for every k

$$\sum_{i=1}^k \gamma_i^2 \leq \sum_{i=1}^{mk} \theta_i^2 \quad \text{as in [DaHa05]} \quad (1)$$

and

$$\gamma_{k+m+1}^2 \leq \sum_{i=k+1}^{k+m} \theta_i^2.$$

[DaHa05] shows that one further inequality completes the list for $\deg(\gamma) \leq 3$, $m = 2$:
 $\gamma_2^2 + \gamma_3^2 \leq \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_6^2$. We can generalize this to (the possibly redundant)

$$\gamma_2^2 + \gamma_3^2 \leq \sum_{i=1}^{2m-1} \theta_i^2 + \theta_{2m+2}^2$$

Feasibility of Pairs — Necessary Inequalities

We can derive classes of necessary inequalities, including:

Lemma

If (γ, θ) is feasible for m , then for every k

$$\sum_{i=1}^k \gamma_i^2 \leq \sum_{i=1}^{mk} \theta_i^2 \quad \text{as in [DaHa05]} \quad (1)$$

and

$$\gamma_{k+m+1}^2 \leq \sum_{i=k+1}^{k+m} \theta_i^2.$$

Which rates $v, w > 0$ of exponential decay are possible based on (1)?:

$$\gamma_i \propto e^{-iv}, \quad \theta_i \propto e^{-iw}$$
$$(\gamma, \theta) \text{ feas. for } m \Rightarrow v \in [\frac{w}{m}, w]$$

Feasibility of Pairs — Linear Programming Approach

- ▶ Let $\gamma^2 = (10, 2, 1, 0.25, 0.25)$, $\theta^2 = (4, 3, 2.5, 2, 2)$

Feasibility of Pairs — Linear Programming Approach

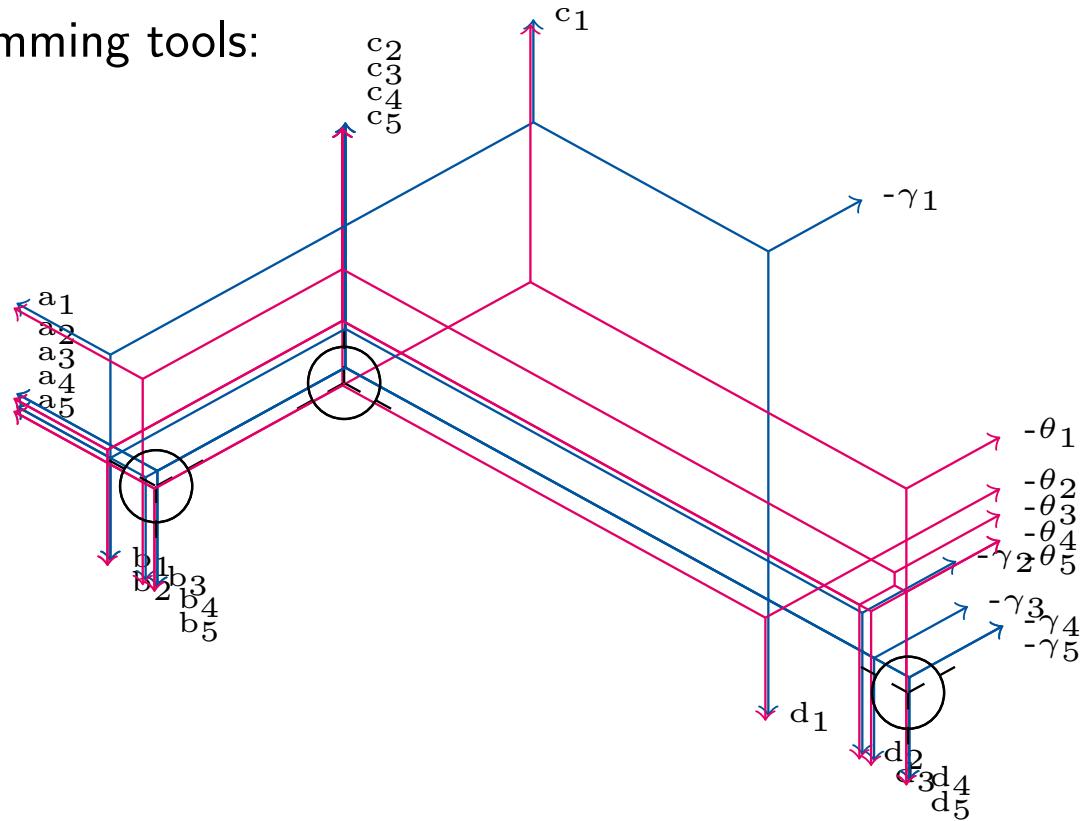
- ▶ Let $\gamma^2 = (10, 2, 1, 0.25, 0.25)$, $\theta^2 = (4, 3, 2.5, 2, 2)$
- ▶ (γ, θ) not feasible for $m \leq 3$ since $10 \not\leq 4 + 3 + 2.5$

Feasibility of Pairs — Linear Programming Approach

- ▶ Let $\gamma^2 = (10, 2, 1, 0.25, 0.25)$, $\theta^2 = (4, 3, 2.5, 2, 2)$
- ▶ (γ, θ) not feasible for $m \leq 3$ since $10 \not\leq 4 + 3 + 2.5$
- ▶ (constructively) feasible for $m = 5$

Feasibility of Pairs — Linear Programming Approach

- Let $\gamma^2 = (10, 2, 1, 0.25, 0.25)$, $\theta^2 = (4, 3, 2.5, 2, 2)$
- (γ, θ) not feasible for $m \leq 3$ since $10 \not\leq 4 + 3 + 2.5$
- (constructively) feasible for $m = 5$
- $m = 4?$ → use linear programming tools:



TT-Feasibility —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

TT-Feasibility

Example

We return to the introduction:

- ▶ $(\sigma^{(\{1\})})^2 = (9, 4.5)$
- ▶ $(\sigma^{(\{1,2\})})^2 = (10, 2, 1, 0.25, 0.25)$
- ▶ $(\sigma^{(\{1,2,3\})})^2 = (4, 3, 2.5, 2, 2)$

Regarding the single pairs:

- ▶ $n_1 = \deg(\sigma^{(\{1\})}) = 2$.
- ▶ $(\sigma^{(\{1\})}, \sigma^{(\{1,2\})})$ is feasible for $n_2 = 3$ as provided by the earlier special case.
- ▶ $(\sigma^{(\{1,2\})}, \sigma^{(\{1,2,3\})})$ is feasible for $n_3 = 4$ as provided by the hive.
- ▶ $n_4 = \deg(\sigma^{(\{1,2,3\})}) = 5$.

Thereby σ is (TT-)feasible for and no less than $n = (2, 3, 4, 5)$.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{\{1, \dots, \mu-1\}}, \sigma^{\{1, \dots, \mu\}})$, $\mu = 2, \dots, d-1$:

Combine constructed nodes $\mathcal{G} = (\mathcal{G}_1, \dots, \mathcal{G}_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ
derive H -descr. (if manageable) ^[DaHa05]	✓	✓	

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ
derive H -descr. (if manageable) ^[DaHa05]	✓	✓	
classes of nec. ineq.		✓	

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ
derive H -descr. (if manageable) ^[DaHa05]	✓	✓	
classes of nec. ineq.		✓	
linear progr. hive (if well conditioned)	✓	✓	if lucky: ✓

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ
derive H -descr. (if manageable) ^[DaHa05]	✓	✓	
classes of nec. ineq.		✓	
linear progr. hive (if well conditioned)	✓	✓	if lucky: ✓
(1) and $(\deg(\gamma) \leq n_\mu \text{ or } \deg(\theta) \leq n_\mu)$ ^[LiPoWa14]	✓	✓	

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ
derive H -descr. (if manageable) ^[DaHa05]	✓	✓	
classes of nec. ineq.		✓	
linear progr. hive (if well conditioned)	✓	✓	if lucky: ✓
(1) and $(\deg(\gamma) \leq n_\mu \text{ or } \deg(\theta) \leq n_\mu)$ ^[LiPoWa14]	✓	✓	
$\deg(\gamma) \leq n_\mu$ and $\deg(\theta) \leq n_\mu$	✓	✓	✓

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1, \dots, \mu-1\})}, \sigma^{(\{1, \dots, \mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ
derive H -descr. (if manageable) ^[DaHa05]	✓	✓	
classes of nec. ineq.		✓	
linear progr. hive (if well conditioned)	✓	✓	if lucky: ✓
(1) and $(\deg(\gamma) \leq n_\mu \text{ or } \deg(\theta) \leq n_\mu)$ ^[LiPoWa14]	✓	✓	
$\deg(\gamma) \leq n_\mu$ and $\deg(\theta) \leq n_\mu$	✓	✓	✓
iteratively enforce sv (if convergent)	✓		✓

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A , $\text{sv}(A) = \sigma$. The construction is possible in parallel.

TT-Feasibility — TT-Feasibility-(Mini)Toolbox

public TT-Feasibility-Toolbox on RWTH Git-Lab Matlab code and practical introduction with live-scripts (also as pdf)

Feasibility of singular values in the Tensor Train format

A 4th-order example

We will go through one example of feasibility for a $d = 4$ -th-order tensor using different algorithms. Let σ_i be as follows:

```
d = 4;
sigma = cell(1,5);
sigma(1) = sqrt(13);
sigma(2) = [3,2];
sigma(3) = [sqrt(10),sqrt(2),1];
sigma(4) = [2,sqrt(3),sqrt(2.5),sqrt(2),sqrt(1.5)];
sigma(5) = sqrt(13);
sigma{};

ans = 3.6056
ans =
    3      2
ans =
   3.1623   1.4142   1.0000
ans =
   2.0000   1.7321   1.5811   1.4142   1.2247
ans = 3.6056
```

1. Using linear programming and hives:

For each pair (σ^0, σ^{d-1}) we can use the linear programming algorithm

`linear_programming_check` construct hives and determine the minimal required mode size n such that the pair is feasible for that value.

```
help linear_programming_check
```

`linear_programming_check` investigate the feasibility of a pair (γ, θ)

`linear_programming_check(n, gamma, theta, drawoptions)` finds the smallest mode size $n \leq n$ for which (γ, θ) is feasible and plots the hive obtained through the linear programming algorithm.

gamma and theta must be arrays of length n but may contain zeros.

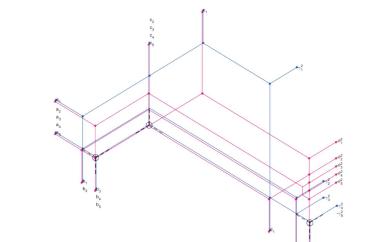
Please note that this code uses honeycomb notation (not tensor notation). n is the size of the hermitian matrices.

Options :

`drawoptions.compact == 1`: less distance between honeycombs
`drawoptions.single == 1`: also draw single honeycombs

Example:

```
gamma = sqrt([10,2,1]);
theta = sqrt([4,3,2.5,2,1.5]);
drawoptions = struct;
drawoptions.compact = 0;
```



Thereby we know that σ is feasible for $n = [2, 2, 4, 5]$ and no less (except if instabilities should have corrupted our results, which is not to be expected for such small examples).

n

2 2 4 5

2. Using the alternating iteration

For each pair we may also use the second algorithm and enforce these singular values iteratively to actually construct a representation. In order to use `alternating_core_generation` we need to specify a suspected η_{ji} .

The cell G now contains a representation of a tensor $A = \pi_7(G)$ which has singular values σ .

Additionally, $G = G$ is the standard representation of A . Without previously finding n , we can also use `construct_tensor_with_sv` to find n and construct all cores in G in parallel.

```
help construct_tensor_with_sv
```

`construct_tensor_with_sv` constructs a tensor with prescribed TT-singular values in parallel

`construct_tensor_with_sv(n, sigma)` expects a starting mode size n with $n(mu) \geq 1$ for all mu and a cell $sigma$, containing TT-singular values, s.t. $\text{length}(\sigma) = \text{length}(n) + 1$, and returns a TT-representation G with $\text{size}(G(mu)) = [r(mu), (mu)+1, n_plus(mu)]$ for the possible increased mode size n_plus and $r(mu) = \text{length}(\sigma(mu))$

`construct_tensor_with_sv(n, sigma, opts)` specifies options for printing and tol

Example (from the talk):

```
sigma = cell(1,5);
sigma(1) = sqrt(13);
sigma(2) = [3,2];
sigma(3) = [sqrt(10),sqrt(2),1];
sigma(4) = [2,sqrt(3),sqrt(2.5),sqrt(2),sqrt(1.5)];
sigma(5) = sqrt(13);
sigma{};

[G,n_plus] = construct_tensor_with_sv(n, sigma)
% returns n_plus = [2,2,4,5]
```

See also:
`test_construct_tensor_with_sv_alternating_core_generation`

We may specify a minimal mode size to start from. We choose $n = 1$ here. The first call may require much more time since Matlab needs to connect to the workers in order to proceed with the `parfor` loop.

```
construct_tensor_with_sv([1,1,1,1], sigma);
```

Starting parallel pool (parpool) using the 'local' profile ...
`sigma` is (numerically) feasible for
 $n = 2 \quad 2 \quad 4 \quad 5$

In this case, the alternating iteration converges in all cases to the desired result. Note that none of the calls of `alternating_core_generation` need to communicate. In this sense, the algorithm runs completely parallel.

3. Diagonal core generation

Based on the Theorem which provides the feasibility of a pair (γ, θ) for $n = \max(\deg(\gamma), \deg(\theta))$, the algorithm `diagonal_core_generation` constructs the required core of length n .

```
G(1) = diagonal_core_generation(sigma(1), sigma(2));
```

Thank you for your attention!

(If interested, just ask or email me for the citations, slides and the current, so far revised paper.)

Feasibility and Compatibility

Reshaping

Tensor Feasibility Problem (TFP)

Feasibility for the Tucker Format

Equivalence of TFP and QMP

Tensor Tree Decompositions / SVDs

Matrix Decompositions

Tensor Decompositions

Decoupling for Hierarchical Tucker

TT and Tucker

Decoupling for TT

Sums of Hermitian Matrices and Honeycombs

Weyl's Problems

Hives

Feasibility of Pairs

Connection to Hives

Necessary Inequalities

Linear Programming Approach

TT-Feasibility

Example from the Introduction

Overview of Methods

TT-Feasibility-(Mini)Toolbox