What to compute?

Part II

Algebraic Geometry

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For geometric applications one of the most important desiderata would be direct images for projection maps $f : Y = X \times Z \to X$ of smooth varieties.

Here we usually work with right $\mathscr{D}$-modules $\mathscr{M}$, so the direct image is

$$\begin{aligned} f_*^{\mathscr{D}}(\mathscr{M}) &= Rf_*(\mathscr{M} \otimes^L_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}) \\ &\simeq Rf_*[\mathscr{M} \otimes_{\mathscr{O}_Y} \mathrm{Alt}^d(\mathscr{T}_{Y/X}) \to \cdots \to \mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{T}_{Y/X} \to \mathscr{M}] \end{aligned}$$

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$$\begin{aligned} * : \quad \mathrm{Hol}(\mathscr{D}_G) \times \mathrm{Hol}(\mathscr{D}_G) \;&\longrightarrow\; \mathrm{D}^b_{hol}(\mathscr{D}_G), \\ \mathscr{M}_1 * \mathscr{M}_2 \;&=\; f_*^{\mathscr{D}}(\mathscr{M}_1 \boxtimes \mathscr{M}_2), \end{aligned}$$

where

- $\mathscr{M}_1 \boxtimes \mathscr{M}_2 = pr_1^*(\mathscr{M}_1) \otimes pr_2^*(\mathscr{M}_2) \in \mathrm{Hol}(\mathscr{D}_{G\times G})$,
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Let's look at two concrete examples: Tori and abelian varieties.

Warm-up: Hypergeometric $\mathscr{D}$-modules in dimension 1

For $P, Q \in \mathbb{C}[s] \setminus \{0\}$ consider the hypergeometric $\mathscr{D}$-module

$$\mathscr{H}_{P,Q} \; = \; \mathscr{D}/\mathscr{D} \cdot (P(z\partial) - zQ(z\partial))$$

on $\mathbb{G}_m = \operatorname{Spec} \mathbb{C}[z, z^{-1}]$.

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- construction of examples in inverse Galois theory,
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In dimension 1 they are all obtained by convolution as follows.

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Any simple holonomic $\mathscr{D}$-module $\mathscr{M}$ on $\mathbb{G}_m$ has non-negative Euler characteristic

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Theorem (Katz)

The following are equivalent:

- $\chi(\mathscr{M}) = 1$,
- $\mathscr{M}$ is hypergeometric,
- $\mathscr{M}$ is an iterated convolution of elementary modules of the form $\mathscr{H}_{c,1} = \delta_c$, $\mathscr{H}_{s-\alpha,1}$ and $\mathscr{H}_{1,s-\alpha}$ where $c \in \mathbb{C}$ and α varies in a set of representatives for $\mathbb{C}/\mathbb{Z}$.

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3. The subcategory $\overline{\mathrm{Hol}}(\mathscr{D}_T) \subset \mathrm{Hol}(\mathscr{D}_T)$ of modules with no subobjects or quotients as above is Tannakian wrt "$*$".

Hypergeometric $\mathscr{D}$-modules on tori

The last phrase means that there is an affine group G over $\mathbb{C}$ with an equivalence

$$\omega : \quad \overline{\mathrm{Hol}}(\mathscr{D}_T) \xrightarrow{\sim} \mathrm{Rep}(G)$$

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Theorem (Loeser-Sabbah)

A module $\mathscr{M} \in \overline{\mathrm{Hol}}(\mathscr{D}_T)$ has $\chi(\mathscr{M}) = 1$ iff it is a convolution of pushforwards of elementary hypergeometric modules on $\mathbb{G}_m$ under closed embeddings $i : \mathbb{G}_m \hookrightarrow T$ of tori.

Remarks

- Euler characteristics have a meaning in algebraic statistics:

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- What about representations of dimension > 1?

  To understand the Tannakian group G one needs to know its invariants in tensor powers of representations, so we want to compute $\mathrm{Hom}(\delta_1, \mathscr{M}_1 * \mathscr{M}_2)$ for $\mathscr{M}_1, \mathscr{M}_2 \in \mathrm{Hol}(\mathscr{D}_T)$.

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  ...this can be done by computer algebra!

Getting serious: Abelian varieties

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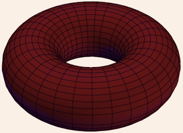

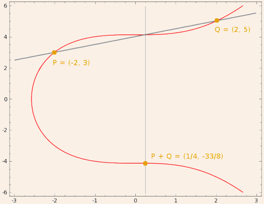

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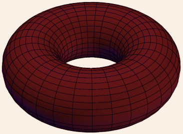

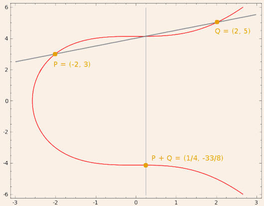

For $g > 1$ it becomes much harder to work with explicit equations for projective models, but there are some well-understood families such as the Horrocks-Mumford abelian surfaces $A \subset \mathbb{P}^4$...

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General Question

For $\mathscr{M} \in \overline{\mathrm{Hol}}(\mathscr{D}_A)$, what is the geometric meaning of the algebraic group

$$G(\mathscr{M}) \ := \ \mathrm{Image}\big(G \to \mathrm{Gl}(\omega(\mathscr{M})\big)?$$

Example: Schottky

If $A = \mathrm{Jac}(C)$ is the Jacobian of a smooth nonhyperelliptic curve C of genus $g > 1$, then the perverse intersection complex on the theta divisor has

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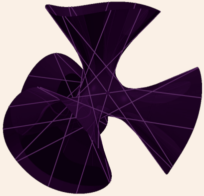
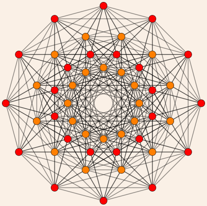
More generally, for any semisimple module $\mathscr{M} \in \overline{\mathrm{Hol}}(\mathscr{D}_A)$ the group $G(\mathscr{M})$ is reductive and the weights of its representations are related to the geometry of the characteristic cycle $\mathrm{CC}(\mathscr{M})$.

Example: Cubic threefolds

Theorem (K)

The intermediate Jacobian $A = \mathrm{Jac}(T)$ of a smooth cubic threefold $T \subset \mathbb{P}^4$ has the Tannakian group $G(\delta_\Theta) \simeq E_6(\mathbb{C})$.





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Can we implement algorithms for $\mathscr{D}$-modules on elliptic curves that are powerful enough to deal with this example?

Let's get started!

Ultimate goal: Given $\mathscr{M}, \mathscr{N} \in \overline{Hol}(\mathscr{D}_A)$, compute $\mathscr{M} * \mathscr{N}$ or at least

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Thank you very much!