Holonomic functions in the field

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ÖAW RICAM

The Holonomic Systems Approach

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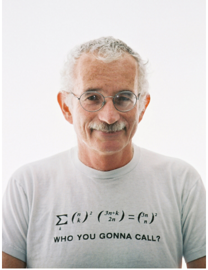
A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.



- seminal paper by Doron Zeilberger in 1990
- created a huge research area
- many applications in mathematics and elsewhere

D-finite and P-recursive

A function $f(x)$ is called **D-finite** if it satisfies a linear ordinary differential equation with polynomial coefficients:

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0,$$

$p_0, \ldots, p_d \in \mathbb{K}[x]$ not all zero.

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A sequence $f(n)$ is called **P-recursive** (or **P-finite**) if it satisfies a linear recurrence equation with polynomial coefficients:

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$\longrightarrow$ Also called **holonomic function** resp. **holonomic sequence**.

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&= \tfrac{2(2n+5)(n^2+5n+5)}{(n+2)(n+3)(n+4)} H_{n+1} - \tfrac{(n+1)(3n^2+18n+26)}{(n+2)(n+3)(n+4)} H_n
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Closure Properties

If $f(x)$ and $g(x)$ are D-finite then also the following are D-finite

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A sequence is P-recursive iff its generating function is D-finite.

Proof

Show that for P-recursive sequences $f(n)$ and $g(n)$ also $h(n) = f(n)g(n)$ is P-recursive. Assume f and g satisfy recurrences of order d_1 and d_2, respectively.

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Ansatz: want to find $c_0, \ldots, c_d \in \mathbb{K}[n]$ such that

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&= \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} r_{i,j}(c_0, \ldots, c_d, n) f(n+i) g(n+j)
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All coefficients $r_{i,j}$ must vanish: this yields $d_1 d_2$ equations for the unknowns $c_0, \ldots, c_d$. The choice $d = d_1 d_2$ ensures a solution.

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Generalize the finiteness property to

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- mixed setting: functions in several continuous and discrete variables $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

$$(x^2 - 1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0.$$

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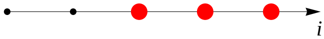

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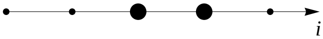

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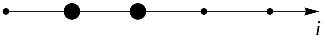

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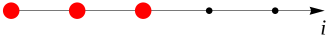

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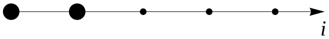

$\longrightarrow$ $P_n(x)$ is D-finite w.r.t. x.

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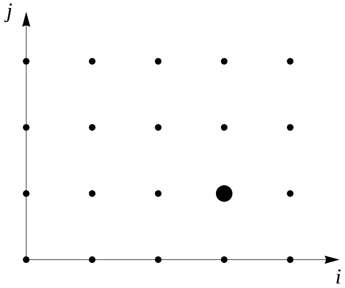

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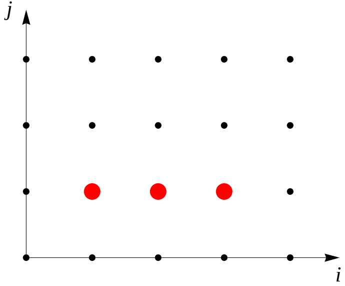

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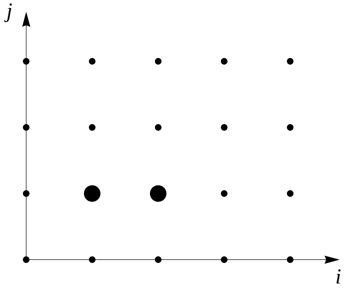

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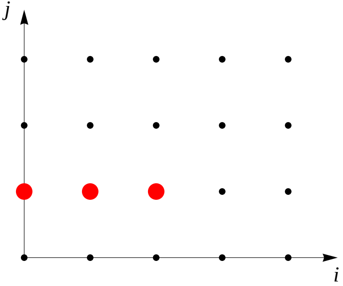

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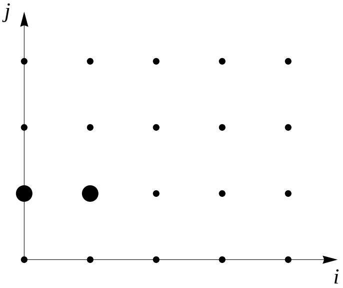

$$P_{n+1}^{(3)}(x) = \frac{(n^2x^2 - n^2 + 3nx^2 - 3n + 8x^2)}{(x^2-1)^2} P_{n+1}'(x) - \frac{4(n^2x + 3nx + 2x)}{(x^2-1)^2} P_{n+1}(x)$$

Example: Legendre Polynomials $P_n(x)$

This family of (orthogonal) polynomials is a particular solution of the differential equation

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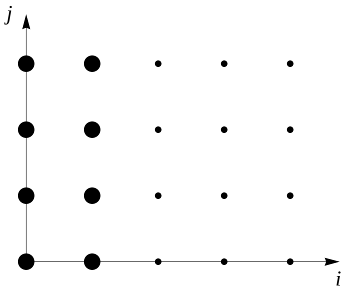

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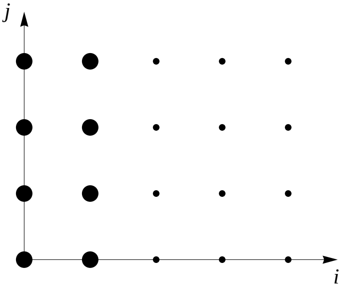

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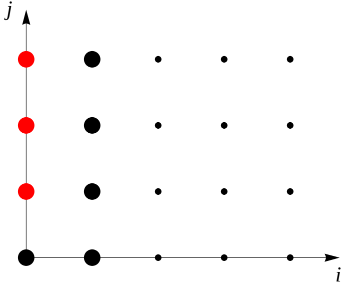

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$\frac{(2n+5)x}{n+3} P_{n+2}(x) - \frac{n+2}{n+3} P_{n+1}(x)$

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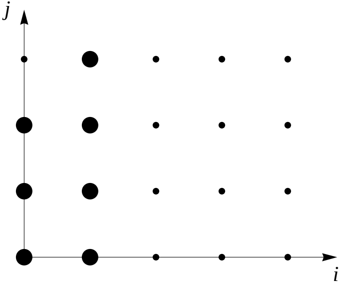

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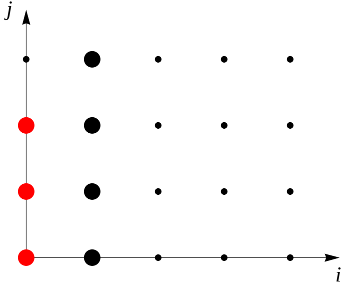

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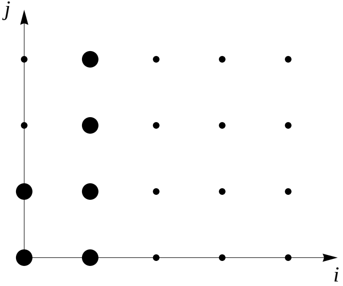

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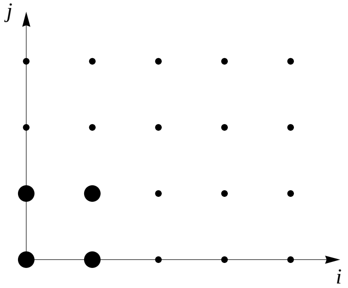

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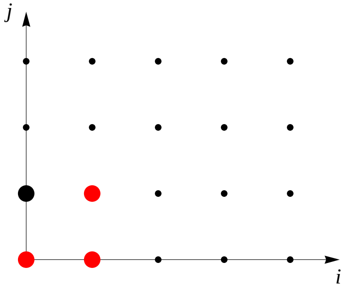

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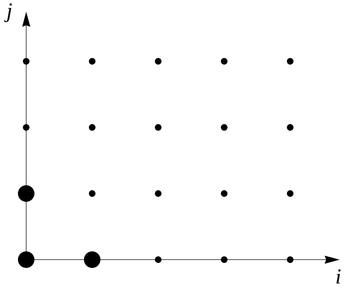

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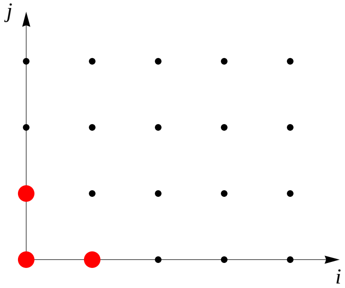

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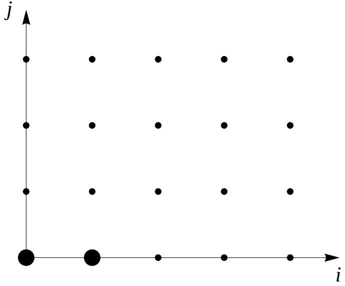

$\longrightarrow P_n(x)$ is ∂-finite w.r.t. n and x (of rank 2).

∂-Finiteness

Let $f(x_1, \ldots, x_s, n_1, \ldots, n_r)$ be a function in the continuous variables $x_1, \ldots, x_s$ and in the discrete variables $n_1, \ldots, n_r$.

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$$\frac{\mathrm{d}^{i_1}}{\mathrm{d}x_1^{i_1}} \cdots \frac{\mathrm{d}^{i_s}}{\mathrm{d}x_s^{i_s}} f(x_1, \ldots, x_s, n_1 + j_1, \ldots, n_r + j_r)$$

with $i_1, \ldots, i_s, j_1, \ldots, j_r \in \mathbb{N}$ such that any shifted partial derivative of f (of the above form) can be expressed as a $\mathbb{K}(x_1, \ldots, x_s, n_1, \ldots, n_r)$-linear combination of the basis functions.

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Again, finitely many initial conditions suffice to specify / fix f.

Algebraic Setting

Write differential/difference equations in operator notation:

- shift operator S_v: $S_v f(v) = f(v+1)$
- partial derivative D_v: $D_v f(v) = \frac{\mathrm{d}}{\mathrm{d}v} f(v)$
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$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

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$$(n+2)S_n^2 - (2n+3)xS_n + (n+1).$$

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Differential equations and recurrences are translated to skew polynomials.

Noncommutative multiplication:

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Even more general:

$$\partial_v a = \sigma(a) \partial_v + \delta(a)$$

where σ is an automorphism and δ a σ-derivation, i.e.,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

Operator Algebra

Such operators form an **Ore algebra**

$$\mathbb{K}(v, w, \dots)\langle\partial_v, \partial_w, \dots\rangle,$$

i.e., multivariate polynomials in the ∂'s with coefficients being rational functions in $v, w, \dots$, where $\mathbb{K}$ is a field, $\mathrm{char}(\mathbb{K}) = 0$.

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Definition: We define the **annihilator** of a function f to be the set

$$\mathrm{Ann}_{\mathbb{O}}\, f := \big\{ P \in \mathbb{O} : P \cdot f = 0 \big\}$$

(it is a **left ideal** in $\mathbb{O}$).

Definition: ∂-Finite Function

Let $\mathbb{O} = \mathbb{K}(v, w, \dots)\langle \partial_v, \partial_w, \dots \rangle$ be an Ore algebra.

A function $f(v, w, \dots)$ is ∂-finite w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f : P \in \mathbb{O}\}$$

form a finite-dimensional $\mathbb{K}(v, w, \dots)$-vector space:

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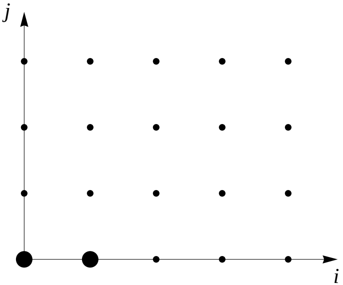

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In other words, if the left ideal of annihilating operators of f

$$\operatorname{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} : P \cdot f = 0\}$$

is a zero-dimensional ideal.

Why ∂-Finite Functions?

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   - multiplication, e.g., $P_n(x)P_{n+1}(x)$
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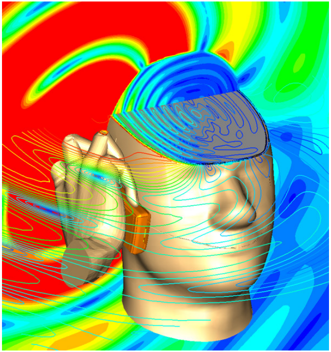
3. These operations (closure properties) can be executed algorithmically.
4. Many elementary and special functions are covered.

(Incomplete) List of ∂-Finite Functions

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, Multinomial, CatalanNumber, QBinomial, CosIntegral, ArcSech, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, Binomial, ParabolicCylinderD, Erfc, EllipticK, Fibonacci, QFactorial, Cos, Hypergeometric2F1, Erf, KelvinKer, HypergeometricPFQRegularized, Log, Factorial, BesselY, Cosh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, BetaRegularized, SphericalHankelH1, ArcSin, EllipticThetaPrime, Root, LucasL, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, HarmonicNumber, BesselI, KelvinKei, ArithmeticGeometricMean, Exp, ArcCot, EllipticTheta, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, Subfactorial, QPochhammer, Gamma, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, BesselJ, StruveL, ArcSec, Factorial2, KelvinBer, BesselK, ArcSinh, HankelH1, Sqrt, PolyGamma, HypergeometricU, AiryAiPrime, Sin,

Finite Elements



Joint work with Joachim Schöberl and Peter Paule

Problem Setting

Simulate the propagation of electromagnetic waves using the Maxwell equations

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (this is the 2D case):

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\tfrac{2y}{1-x} - 1\right)$$

using the Legendre and Jacobi polynomials.

Problem: Represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself).

Make an Ansatz!

More precisely, we need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) \;=\; \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\tfrac{\mathrm{d}}{\mathrm{d}y}$).

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that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

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7. If there is no solution, go back to step 3.

Result

With this method, we find the relation

$$\begin{aligned}
&(2i+j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\
&2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\
&(j+3)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\
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$\longrightarrow$ The use of these previously unknown formulae caused a considerable speed-up in the numerical simulations.

Symbolic Summation and Integration

That was nice, but we want (and can) do more...

What about integrals

$$\int_a^b f(x, \dots)\,\mathrm{d}x$$

and sums

$$\sum_{n=a}^{b} f(n, \dots)$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

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Then $F(n) = \sum_{k=a}^{b} \big(g(n,k+1) - g(n,k)\big) = g(n,b+1) - g(n,a)$.

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Creative Telescoping: write

$$c_d(n) f(n+d,k) + \cdots + c_0(n) f(n,k) = g(n,k+1) - g(n,k).$$

Summing from a to b yields a recurrence for $F(n)$:

$$c_d(n) F(n+d) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

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$$c_d(x) \frac{\mathrm{d}^d}{\mathrm{d}x^d} f(x,y) + \cdots + c_0(x) f(x,y) \; = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_d(x) \frac{\mathrm{d}^d}{\mathrm{d}x^d} F(x) + \cdots + c_0(x) F(x) \; = g(x,b) - g(x,a)$$

Creative Telescoping, $\mathbb{O} = \mathbb{K}(n,k)\langle S_n, S_k\rangle$

$$\begin{aligned} c_d(n)f(n+d,k) + \cdots + c_0(n)f(n,k) &= g(n,k+1) - g(n,k) \\ &= (S_k - 1)\cdot g(n,k). \end{aligned}$$

Where should we look for a suitable $g(n,k)$?
Note that there are trivial solutions like:

$$g(n,k) := \sum_{i=0}^{k-1} \big(c_d(n)f(n+d,i) + \cdots + c_0(n)f(n,i)\big)$$

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Then the task is to find $P(n,S_n) = c_d(n)S_n^d + \cdots + c_0(n)$ and $Q \in \mathbb{O}$ such that

$$\big(P - (S_k - 1)Q\big)\cdot f = 0 \quad \iff \quad P - (S_k - 1)Q \in \operatorname{Ann}_{\mathbb{O}}(f).$$

Creative Telescoping (Example 1)

Let $F(n)$ denote the double sum over the trinomial coefficients

$$F(n) = \sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i, j, n-i-j} = \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{n!}{i!j!(n-i-j)!}.$$

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Then the creative telescoping operator

$$CT = S_n - 3 + (S_i - 1)\frac{i}{n-i-j+1} + (S_j - 1)\frac{j}{n-i-j+1}$$

with $CT\left(\binom{n}{i,j,n-i-j}\right) = 0$ implies that

$$F(n+1) = 3F(n).$$

Creative Telescoping (Example 2)

The lattice Green's function of the square lattice is given by

$$G(z) = \int_0^1 \int_0^1 \frac{1}{(1-xyz)\sqrt{1-x^2}\sqrt{1-y^2}} \, \mathrm{d}x \, \mathrm{d}y.$$

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The creative telescoping operator

$$(z^3 - z)D_z^2 + (3z^2 - 1)D_z + z + D_x \frac{y(1-x^2)}{xyz-1} + D_y \frac{yz(1-y^2)}{xyz-1}$$

that annihilates the integrand, certifies that $G(z)$ satisfies the differential equation

$$(z^3 - z)G''(z) + (3z^2 - 1)G'(z) + zG(z) = 0.$$

How to Find (P, Q)?

Make an ansatz for the telescoper P and the certificate Q.

Fix an integer r and set

$$P = \sum_{i=0}^{r} p_i(x) D_x^i \qquad \text{with } p_i \in \mathbb{K}(x) \text{ unknown coefficients.}$$

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Let $\mathfrak{U}$ denote the set of monomials under the stairs of a Gröbner basis for $\operatorname{Ann}_{\mathbb{O}}(f)$, or any other vector space basis of $\mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)$.

Since $Q \in \mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)$, we can set

$$Q = \sum_{u \in \mathfrak{U}} q_u(x, y)\, u \qquad \text{with unknown } q_u \in \mathbb{K}(x, y).$$

Chyzak's Algorithm

Putting things together:

$$P - D_y Q = \sum_{i=0}^{r} p_i(x) D_x^i - D_y \sum_{u \in \mathfrak{U}} q_u(x,y)\, u$$

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Since we want $P - D_y Q \in \mathrm{Ann}_{\mathbb{O}}(f)$ we reduce the above expression with a Gröbner basis of $\mathrm{Ann}_{\mathbb{O}}(f)$ and equate the (D_x, D_y)-coefficients to zero.

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Finally: loop over the (a priori) unknown order r of the telescoper.
$\longrightarrow$ This is Chyzak's algorithm (analogously in other Ore algebras).

Creative Telescoping in Full Generality

In general, a creative telescoping operator has the form

$$P(\boldsymbol{v}, \boldsymbol{\partial_v}) + \Delta_1 Q_1(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w}) + \cdots + \Delta_m Q_m(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\partial_v}, \boldsymbol{\partial_w})$$

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- Research topic: develop fast algorithms to compute it!

Ansatz with Specific Denominators

For finding CT operators, we proposed an ansatz of the form

$$\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(\boldsymbol{v})\boldsymbol{\partial}_{\boldsymbol{v}}^{\boldsymbol{\alpha}} + \sum_{i=1}^{m} \Delta_i \sum_{u\in\mathfrak{U}} \frac{\sum_{\boldsymbol{\beta}} q_{i,j,\boldsymbol{\beta}}(\boldsymbol{v})\boldsymbol{w}^{\boldsymbol{\beta}}}{d_{i,j}(\boldsymbol{v},\boldsymbol{w})}\, u$$

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- implemented in `HolonomicFunctions` (Mathematica)

Reduction-Based Telescoping

- Typically, the certificate Q is much larger than the telescoper.
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If the h_i live in a finite-dimensional $\mathbb{K}(x)$-vector space, then there exists a nontrivial linear combination $p_0 h_0 + \ldots + p_r h_r = 0$.

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$\longrightarrow$ Hence, the desired telescoper is $p_0 + p_1 D_x + \ldots + p_r D_x^r$.

Holonomy

Question: Does there always exist such a telescoping operator?

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Combine the two notions:

- Use ∂-finiteness for computations.
- Use holonomy for justifications.

Holonomic Functions

Assume that $f(x_1, \ldots, x_s)$ depends only on continuous variables. Consider the **Weyl algebra**

$$\mathbb{W} = \mathbb{K}[x_1, \ldots, x_s]\langle D_{x_1}, \ldots, D_{x_s}\rangle.$$

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Differently stated: f is holonomic if for any $(s-1)$-subset

$$E \subset \{x_1, \ldots, x_s, D_{x_1}, \ldots, D_{x_s}\}, \quad |E| = s - 1,$$

there exists a nonzero element in $\mathrm{Ann}_{\mathbb{W}}(f)$ that is free of all generators in E.

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$\longrightarrow$ This is why a creative telescoping operator always exists.

∂-Finite and Holonomic Functions

Theorem: The function $f(x_1, \ldots, x_s)$ is holonomic if and only if it is ∂-finite.

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Example: The sequence $\frac{1}{n^2+k^2}$ is ∂-finite but not holonomic.

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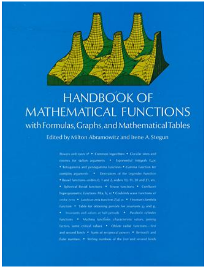
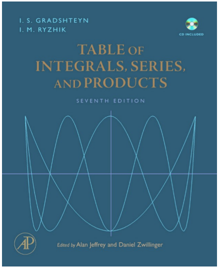
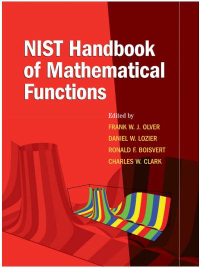
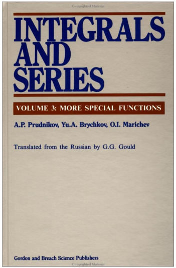
4. The output is always given as an annihilating ideal, not as a closed form.

Special Function Identities









Some Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad (1)$$

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi P_m^{(m+\frac{1}{2},-m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \qquad (2)$$

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\left(2\sqrt{tx}\right)\mathrm{d}t \qquad (3)$$

$$\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)r^m s^n e^{-x^2}}{m!n!}\,\mathrm{d}x = \sqrt{\pi}e^{2rs} \qquad (4)$$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1}a^\nu n!\Gamma(\nu)} \qquad (5)$$

$$\frac{\sin\left(\sqrt{z^2+2tz}\right)}{z} = \sum_{n=0}^{\infty} \frac{(-t)^n y_{n-1}(z)}{n!} \qquad (6)$$

Computer Proof of a Special Function Identity

$$e^{-x}x^{a/2}n!L_n^a(x) = \int_0^\infty e^{-t}t^{\frac{a}{2}+n}J_a\big(2\sqrt{tx}\big)\,\mathrm{d}t.$$

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```
<< RISC`HolonomicFunctions`
Annihilator[Exp[-x]*x^(a/2)*n!*LaguerreL[n, a, x],
            {S[a], S[n], Der[x]}]
```

$$\{2S_n - 2xD_x + (-a-2n-2),$$
$$4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),$$
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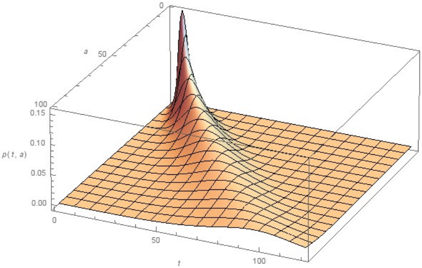
```
CreativeTelescoping[Exp[-t]*t^(a/2+n)*BesselJ[a,2*Sqrt[t*x]],
                    Der[t], {S[a], S[n], Der[x]}]
```

$$\{\{-2S_n + 2xD_x + (a+2n+2),\\ 4x^2D_x^2 + (4x^2+4x)D_x + (-a^2+2ax+4nx+4x),\\ 2xS_a^2 + (2ax+2x^2+2x)D_x + (-a^2+ax-a+2nx+2x)\},\\ \{-2t, -4tx, -2tx\}\}$$

$\longrightarrow$ The annihilating ideals agree; check a few initial values.

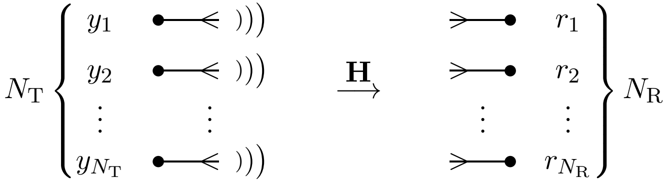
MIMO Wireless Communication Systems



Joint work with Constantin Siriteanu, Akimichi Takemura, Satoshi Kuriki, Donald St. P. Richards, Hyundong Shin

MIMO Wireless Communication Systems

MIMO = Multiple Input + Multiple Output:



Notation:

- N_T: number of transmitting antennas
- N_R: number of receiving antennas
- $\mathbf{y} = (y_1, y_2, \ldots, y_{N_T})^{\mathcal{T}} \in \mathbb{C}^{N_T}$: transmitted signal vector
- $\mathbf{H}$: the $N_R \times N_T$ channel matrix
- $\mathbf{r} = (r_1, r_2, \ldots, r_{N_R})^{\mathcal{T}} = \mathbf{Hy} + \mathbf{n}$: received signal vector, where $\mathbf{n}$ is some additive zero-mean Gaussian noise

Channel Matrix

The channel matrix is modeled as a complex-valued Gaussian random matrix, written as

$$\mathbf{H} = \mathbf{H}_{\mathrm{d}} + \mathbf{H}_{\mathrm{r}}$$

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Fading:

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For sake of simplicity (not w.l.o.g.!), certain assumptions on $\mathbf{H}$:

- $\mathbf{H}_{\mathrm{d}}$ has rank 1
- further assumptions (zero row correlation, etc.)

Zero-Forcing Detection

Recall:

$$\mathbf{r} = \mathbf{H}\mathbf{y} + \mathbf{n}.$$

Zero-Forcing means finding the (modulation constellation) symbols closest to each element of vector

$$\left(\mathbf{H}^{\mathcal{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{r} = \mathbf{y} + \left(\mathbf{H}^{\mathcal{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{n}.$$

Goal of the analysis: say something about the quality of the connection, i.e., how many symbols are transmitted correctly in average.

The following parameters will be used:

- $N = N_{\mathrm{R}} - N_{\mathrm{T}} + 1$
- x_1, x_2: related to $\|\mathbf{H}_{\mathrm{d}}\|^2/\mathbb{E}\{\|\mathbf{H}_{\mathrm{r}}\|^2\}$
- Γ_1: related to the additive noise

Signal-to-Noise Ratio (SNR)

The SNR is the ultimate performance measure (determines the quality of the connection).

Theorem. The moment generating function $M(s; x_1, x_2)$ of the SNR for zero-forcing under full-Rician fading with $r = 1$ is

$$M(s; x_1, x_2) = \frac{e^{-x_2}}{(1 - \Gamma_1 s)^N} \sum_{n_2=0}^{\infty} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s x_1}{1 - \Gamma_1 s}\right).$$

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Definition. The **hypergeometric function** ${}_1F_1$ is defined by

$$ {}_1F_1(a; b; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \quad \text{where}$$

$$(a)_k := a \cdot (a+1) \cdots (a+k-1), \quad (a)_0 := 1$$

is the **Pochhammer symbol** (or rising factorial).

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Theorem. The moment generating function $M(s; x_1, x_2)$ of the SNR for zero-forcing under full-Rician fading with $r = 1$ is

$$M(s; x_1, x_2) = \frac{e^{-x_2}}{(1 - \Gamma_1 s)^N} \sum_{n_2=0}^{\infty} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s x_1}{1 - \Gamma_1 s}\right) =$$

$$e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1}}{(1 - s\Gamma_1)^{N + n_1 - m_1}}.$$

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Obtain the SNR probability density function by Laplace transform:

$$\frac{1}{(1-s\Gamma_1)^{N+n_1-m_1}} \quad \overset{\text{Laplace}}{\longleftrightarrow} \quad \frac{t^{N+n_1-m_1-1} e^{-t/\Gamma_1}}{(N+n_1-m_1-1)!\, \Gamma_1^{N+n_1-m_1}}$$

SNR Probability Density Function

Thus we obtain for the SNR probability density function $p(t; x_1, x_2)$:

$$p(t; x_1, x_2) = \int_0^\infty e^{-st} M(s; x_1, x_2)\, \mathrm{d}s$$

$$= e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!}$$

$$\times \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1} t^{N+n_1-m_1-1} e^{-t/\Gamma_1}}{(N + n_1 - m_1 - 1)!\, \Gamma_1^{N+n_1-m_1}}.$$

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Thus we obtain for the SNR probability density function $p(t; x_1, x_2)$:

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$$= e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!}$$
$$\times \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1} t^{N+n_1-m_1-1} e^{-t/\Gamma_1}}{(N+n_1-m_1-1)!\,\Gamma_1^{N+n_1-m_1}}.$$

Definition. Using this, we define the main object of interest, the **outage probability** $P_{\mathrm{o}}(x_1, x_2)$:

$$P_{\mathrm{o}}(x_1, x_2) = \int_0^\tau p(t; x_1, x_2)\,\mathrm{d}t$$

where τ is a certain prescribed SNR threshold.

Evaluate

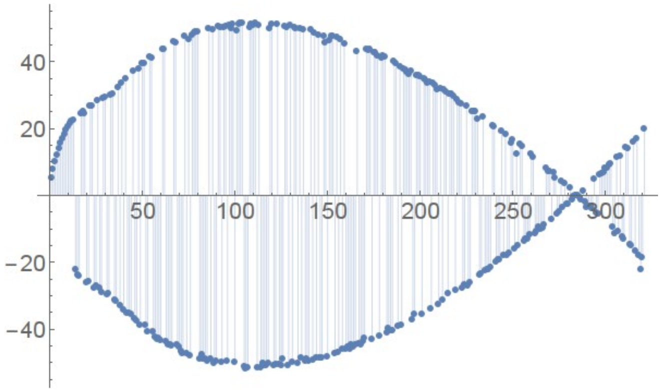
Now, for certain choices of the parameters $N_{\mathrm{R}}, N, x_1, x_2, \Gamma_1, \tau$, we want to "compute" (i.e., evaluate numerically) the outage probability.

First try: truncate the infinite series

$$P_{\mathrm{o}}(x_1, x_2) = e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \times \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1} \gamma(N + n_1 - m_1, \tau/\Gamma_1)}{(N + n_1 - m_1 - 1)!}$$

$\longrightarrow$ Problem: slow convergence.

Difficulties in the Evaluation



- Accuracy problems with standard floating-point arithmetic.

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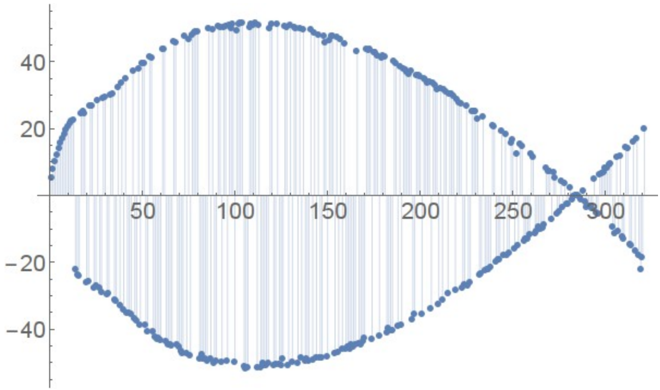

- Accuracy problems with standard floating-point arithmetic.
- Use arbitrary-precision in a computer algebra system.
  But this makes computations even slower.

Holonomic Gradient Method (HGM)

$\longrightarrow$ Methods for evaluating and optimizing certain expressions.
(Nakayama, Nishiyama, Noro, Ohara, Sei, Takayama, Takemura)

Input: $f(x_1, \ldots, x_s)$ holonomic, $(a_1, \ldots, a_s) \in \mathbb{R}^s$
Output: an approximation of $f(a_1, \ldots, a_s)$

1. Determine a holonomic system (set of differential equations) to which f is a solution, and let r be its holonomic rank.
2. Determine a suitable "basis" of derivatives $\mathbf{f} = \left(f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}\right)$ of $f(x_1, \ldots, x_s)$.
3. Convert the holonomic system into a set of Pfaffian systems, i.e., $\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbf{f} = \mathbf{A}_i\mathbf{f}$ for each x_i.
4. Compute $f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}$ at a suitably chosen point $(b_1, \ldots, b_s) \in \mathbb{R}^s$, for which this is easy to achieve.
5. Use your favourite numerical integration procedure (e.g., Euler, Runge-Kutta) to obtain $\mathbf{f}(a_1, \ldots, a_s)$.

Closure Properties (Example)

We have seen that the following expression is holonomic:

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$$ {}_1F_1\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s}{1 - \Gamma_1 s x_1}\right) $$

Substitution $a \to N, b \to n_2 + N_{\mathrm{R}}, x \to \dfrac{\Gamma_1 s}{1 - \Gamma_1 s x_1}$

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$\dfrac{x_2^{n_2}}{n_2!}$ is holonomic (the generating function is $e^{x_2 y}$).

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$$\frac{x_2^{n_2}}{n_2!}\,{}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s}{1 - \Gamma_1 s x_1}\right)$$

Multiplication

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Summation

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e^{-x_2} is holonomic.

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$(1-\Gamma_1 s)^N$ is holonomic.

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Division

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$(1-\Gamma_1 s)^{-N}$ is holonomic as well!

Closure Properties (Example)

We have seen that the following expression is holonomic:

$$M(s; x_1, x_2) = \frac{e^{-x_2}}{(1-\Gamma_1 s)^N} \sum_{n_2=0}^{\infty} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s}{1 - \Gamma_1 s x_1}\right)$$

Hence, by inspection, our SNR moment generating function is holonomic. Likewise, $p(t; x_1, x_2)$ and $P_{\mathrm{o}}(x_1, x_2)$ are holonomic.

Pfaffian Systems

Fix $f(x_1, \ldots, x_s)$.

A suitable "basis of derivatives" $\mathbf{f} = \left(f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}\right)$ for HGM step 2 is given by the (finite!) list of monomials that are irreducible modulo the annihilator ideal.

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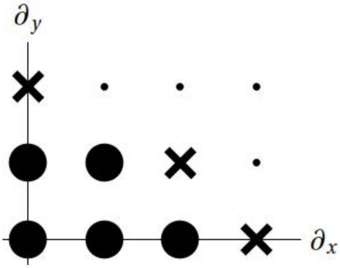

"monomials under the staircase" $(r = 5)$

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The **Pfaffian system** (given by the matrix $\mathbf{A}_i$) for x_i

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbf{f} = \mathbf{A}_i\mathbf{f}$$

is obtained by reduction with the Gröbner basis.

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Nota bene. For $s = 1$ (ODE case) the matrix $\mathbf{A}$ is a companion matrix.

Annihilator for $M(s; x_1, x_2)$

Apply creative telescoping (HolonomicFunctions package) to

$$\sum_{n_2=0}^{\infty} \frac{e^{-x_2}}{(1-\Gamma_1 s)^N} \frac{x_2^{n_2}}{n_2!} \, {}_1F_1\!\left(N; n_2 + N_{\mathrm{R}}; \frac{\Gamma_1 s x_1}{1-\Gamma_1 s}\right)$$

```
annM =
 CreativeTelescoping[Exp[-x2] / (1 - G1 * s) ^ N * x2 ^ n2 / n2 ! *
    Hypergeometric1F1[N, n2 + NR, G1 * s * x1 / (1 - G1 * s)],
   S[n2] - 1, {Der[s], Der[x1], Der[x2]}][[1]]
```

$$\{(-s + G1\, s^2)\, D_s + x1\, D_{x1} + G1\, N\, s,$$
$$(-G1\, s\, x1\, x2 + x2^2 - G1\, s\, x2^2)\, D_{x2}^2 + (-NR\, x1 + G1\, NR\, s\, x1)\, D_{x1} +$$
$$(G1\, N\, s\, x1 - G1\, NR\, s\, x1 + NR\, x2 - G1\, NR\, s\, x2 - G1\, s\, x1\, x2 + x2^2 - G1\, s\, x2^2)$$
$$D_{x2} + G1\, N\, s\, x1,\ (G1\, s\, x1 - x2 + G1\, s\, x2)\, D_{x1}\, D_{x2} +$$
$$(-NR + G1\, NR\, s + G1\, s\, x1 - x2 + G1\, s\, x2)\, D_{x1} + G1\, N\, s\, D_{x2} + G1\, N\, s,$$
$$(G1\, s\, x1^2 - G1^2\, s^2\, x1^2 - x1\, x2 + 2\, G1\, s\, x1\, x2 - G1^2\, s^2\, x1\, x2)\, D_{x1}^2 +$$
$$(G1\, NR\, s\, x1 - G1^2\, NR\, s^2\, x1 - G1^2\, s^2\, x1^2 + G1\, s\, x1\, x2 - G1^2\, s^2\, x1\, x2)\, D_{x1} +$$
$$(-G1\, N\, s\, x2 + G1^2\, N\, s^2\, x2)\, D_{x2} - G1^2\, N\, s^2\, x1\}$$

Annihilator for $p(t; x_1, x_2)$

```
ops = {Der[s], Der[t], Der[x1], Der[x2]};
annM1 = ToOrePolynomial[Prepend[annM, Der[t]], OreAlgebra @@ ops];
annp = CreativeTelescoping[
  DFiniteTimes[annM1, Annihilator[Exp[-s * t], ops]], Der[s]][[1]]
```

$$
\begin{aligned}
\{ & \left(G1\, x1^2\, x2 + 2\, G1\, x1\, x2^2 + G1\, x2^3\right) D_{x2}^2 + \\
& G1\, NR\, t\, x1\, D_t + \left(-G1\, NR\, x1^2 - G1\, NR\, x1\, x2\right) D_{x1} + \\
& \left(-G1\, N\, x1^2 + G1\, NR\, x1^2 - G1\, N\, x1\, x2 + 2\, G1\, NR\, x1\, x2 + \right. \\
& \quad \left. t\, x1\, x2 + G1\, x1^2\, x2 + G1\, NR\, x2^2 + 2\, G1\, x1\, x2^2 + G1\, x2^3\right) D_{x2} + \\
& \left(G1\, NR\, x1 - G1\, N\, NR\, x1 + NR\, t\, x1 - G1\, N\, x1^2 - G1\, N\, x1\, x2 + t\, x1\, x2\right), \\
& \left(-G1\, x1^2 - 2\, G1\, x1\, x2 - G1\, x2^2\right) D_{x1}\, D_{x2} + G1\, NR\, t\, D_t + \\
& \left(-G1\, NR\, x1 - G1\, x1^2 - G1\, NR\, x2 - 2\, G1\, x1\, x2 - G1\, x2^2\right) D_{x1} + \\
& \left(-G1\, N\, x1 - G1\, N\, x2 + t\, x2\right) D_{x2} + \\
& \left(G1\, NR - G1\, N\, NR + NR\, t - G1\, N\, x1 - G1\, N\, x2 + t\, x2\right), \\
& \left(G1\, x1^3 + 2\, G1\, x1^2\, x2 + G1\, x1\, x2^2\right) D_{x1}^2 + \\
& \left(G1\, t\, x1^2 + G1\, NR\, t\, x2 + 2\, G1\, t\, x1\, x2 + G1\, t\, x2^2\right) D_t + \\
& \left(G1\, NR\, x1^2 + G1\, x1^3 + G1\, NR\, x1\, x2 + 2\, G1\, x1^2\, x2 + G1\, x1\, x2^2\right) D_{x1} + \\
& \left(-G1\, N\, x1\, x2 - G1\, N\, x2^2 + t\, x2^2\right) D_{x2} + \left(G1\, x1^2 + G1\, NR\, x2 - G1\, N\, NR\, x2 + \right. \\
& \quad \vdots
\end{aligned}
$$

Annihilator for $P_o(x_1, x_2)$

Recall:

$$P_o(x_1, x_2) = \int_0^\tau p(t; x_1, x_2)\, dt$$

Hence we apply creative telescoping to $p(t; x_1, x_2)$:

```
ct = CreativeTelescoping[annp, Der[t]]
```

$$\left\{ \{D_{x2},\ D_{x1}\},\ \left\{ \frac{G1\ N\ t - t^2}{N\ x1} D_t + \frac{t}{N} D_{x1} - \frac{t}{N} D_{x2} + \frac{G1^2\ N - G1^2\ N^2 - G1\ t + 2\ G1\ N\ t - t^2}{G1\ N\ x1},\ \frac{G1\ t}{x1} D_t + \frac{G1 - G1\ N + t}{x1} \right\} \right\}$$

Annihilator for $P_o(x_1, x_2)$

```
OreGroebnerBasis[
 Flatten[
  MapThread[Function[{p, q},
    (# ** p) & /@ DFiniteSubstitute[DFiniteOreAction[annp, q],
      {t → τ}, Algebra → OreAlgebra[Der[x1], Der[x2]]]], ct]]
```

$$\{-x1\, D_{x1} D_{x2} - x2\, D_{x2}^2 - x1\, D_{x1} + (-NR - x2)\, D_{x2},$$
$$(G1\, x1^2\, x2 + 2\, G1\, x1\, x2^2 + G1\, x2^3)\, D_{x2}^3 + G1\, NR\, x1^2\, D_{x1}^2 +$$
$$(G1\, x1^2 - G1\, N\, x1^2 + G1\, NR\, x1^2 + 3\, G1\, x1\, x2 - G1\, N\, x1\, x2 + 4\, G1\, NR\, x1\, x2 +$$
$$G1\, x1^2\, x2 + 2\, G1\, x2^2 + 2\, G1\, NR\, x2^2 + 2\, G1\, x1\, x2^2 + G1\, x2^3 + x1\, x2\, \tau)\, D_{x2}^2 +$$
$$(2\, G1\, NR\, x1^2 + G1\, NR\, x1\, x2)\, D_{x1} + (G1\, NR\, x1 - G1\, N\, NR\, x1 + 2\, G1\, NR^2\, x1 +$$
$$G1\, x1^2 - G1\, N\, x1^2 + G1\, NR\, x2 + G1\, NR^2\, x2 + 3\, G1\, x1\, x2 - G1\, N\, x1\, x2 +$$
$$2\, G1\, NR\, x1\, x2 + 2\, G1\, x2^2 + G1\, NR\, x2^2 + NR\, x1\, \tau + x1\, x2\, \tau)\, D_{x2},$$
$$(-G1\, x1^4 - 2\, G1\, x1^3\, x2 - G1\, x1^2\, x2^2)\, D_{x1}^3 + (-G1\, x1^3 - G1\, NR\, x1^3 - 2\, G1\, x1^4 -$$
$$2\, G1\, x1^2\, x2 - 2\, G1\, NR\, x1^2\, x2 - 4\, G1\, x1^3\, x2 - G1\, x1\, x2^2 - 2\, G1\, x1^2\, x2^2)\, D_{x1}^2 +$$
$$(-G1\, x1^3\, x2 - G1\, x1\, x2^2 - G1\, N\, x1\, x2^2 - G1\, NR\, x1\, x2^2 -$$
$$2\, G1\, x1^2\, x2^2 - G1\, x2^3 - G1\, N\, x2^3 - G1\, x1\, x2^3 + x2^3\, \tau)\, D_{x2}^2 +$$
$$(-G1\, x1^3 - G1\, N\, x1^3 - G1\, x1^4 - 2\, G1\, x1^2\, x2 - 2\, G1\, N\, x1^2\, x2 -$$

⋮

HGM computation

The irreducible monomials of the annihilator of $P_{\rm o}(x_1, x_2)$ are

$$1, D_1, D_2, D_1^2, D_2^2.$$

Hence, we take the following basis:

$$\mathbf{f} = \left(P_{\rm o}, P_{\rm o}^{(0,1)}, P_{\rm o}^{(1,0)}, P_{\rm o}^{(2,0)}, P_{\rm o}^{(0,2)}\right).$$

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The matrix $\mathbf{A}_1$ of the Pfaffian system $D_1\mathbf{f} = \mathbf{A}_1\mathbf{f}$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{N_{\mathrm{R}}+x_2}{x_1} & -1 & -\frac{x_2}{x_1} & 0 \\ 0 & \langle\cdots\rangle & -\frac{N_{\mathrm{R}}x_1(2x_1+x_2)}{x_2(x_1+x_2)^2} & \langle\cdots\rangle & -\frac{N_{\mathrm{R}}x_1^2}{x_2(x_1+x_2)^2} \\ 0 & \langle\cdots\rangle & \frac{N_{\mathrm{R}}x_1}{(x_1+x_2)^2} & \langle\cdots\rangle & -\frac{(x_1+x_2)^2+N_{\mathrm{R}}x_2}{(x_1+x_2)^2} \end{pmatrix}.$$

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Similar for $D_2\mathbf{f} = \mathbf{A}_2\mathbf{f}$.

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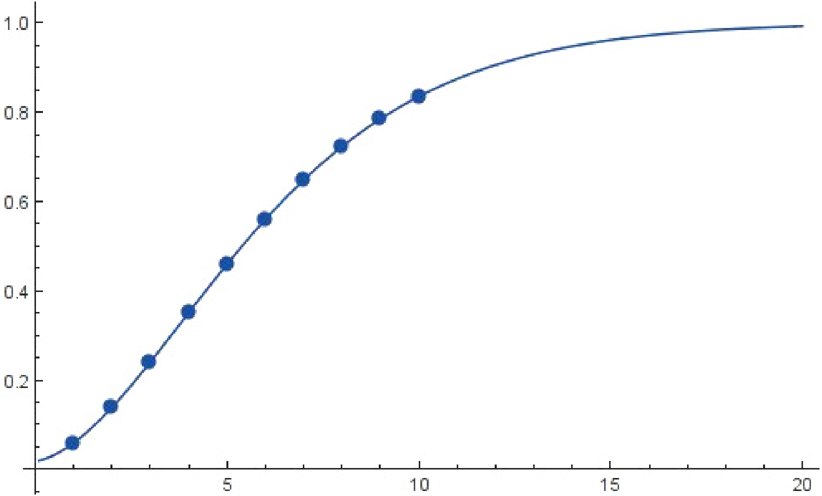
$\mathbf{A}_1$ and $\mathbf{A}_2$ allow to propagate the initial values along both coordinate axes.

HGM computation



- dots: computed with truncated series $(167s)$
- line: computed with HGM $(< 1s)$