Ehrenpreis–Palamodov Theorem
(actually, Noetherian operators and primary ideals)

Yairon Cid Ruiz

Max Planck Institute for Mathematics in the Sciences

A little bit of history and context

In 1960, Ehrenpreis announced his Fundamental Principle, which states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals.

A little bit of history and context

In 1960, Ehrenpreis announced his Fundamental Principle, which states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals.

But, there was a mistake in his proof. Palamodov pointed out the mistake.

A little bit of history and context

In 1960, Ehrenpreis announced his Fundamental Principle, which states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals.

But, there was a mistake in his proof. Palamodov pointed out the mistake.

Then, complete and correct proofs were presented (independently) by Ehrenpreis and Palamodov.

A little bit of history and context

In 1960, Ehrenpreis announced his Fundamental Principle, which states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals.

But, there was a mistake in his proof. Palamodov pointed out the mistake.

Then, complete and correct proofs were presented (independently) by Ehrenpreis and Palamodov.

Later, other slightly different proofs were given by Björk and Hörmander.

The Theorem

Let $I = (p_1, \ldots, p_m) \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$ and $\mathcal{K} \subset \mathbb{R}^n$ be a compact convex set. We consider the system of equations:

$$p_1(i\partial) \cdot u = 0,\ p_2(i\partial) \cdot u = 0,\ \ldots,\ p_m(i\partial) \cdot u = 0,$$
$$u \in C^\infty(\mathcal{K}).$$

The Theorem

Let $I = (p_1, \ldots, p_m) \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$ and $\mathcal{K} \subset \mathbb{R}^n$ be a compact convex set. We consider the system of equations:

$$p_1(i\partial) \cdot u = 0, \; p_2(i\partial) \cdot u = 0, \; \ldots, \; p_m(i\partial) \cdot u = 0,$$
$$u \in C^\infty(\mathcal{K}).$$

Ehrenpreis–Palamodov Theorem

There exist polynomials $\{A_j(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)\}_{j=1}^t$ (**independent of** $\mathcal{K}$) s.t.: If $u \in C^\infty(\mathcal{K})$ is a solution of the above system, then there exist measures $\{\mu_j\}_{j=1}^t$ on a variety $V_j \subset \mathbb{C}^n$ that corresponds with an associated prime of I, s.t.:

$$u(x) = \sum_{j=1}^{t} \int_{V_j} A_j(x, \xi) \exp\left(-i\langle x, \xi \rangle\right) d\mu_j(\xi).$$

What is so special about these polynomials $A_j(x, \xi)$?

Make the substitutions $\xi_i \mapsto \partial_i$ and $\partial_i \mapsto x_i$:

$$A_j(x, \xi) \mapsto A_j(x, \partial) \in \mathbb{C}\langle x, \partial \rangle \quad \text{and} \quad p_k(\partial) \mapsto p_k(x) \in \mathbb{C}[x].$$

What is so special about these polynomials $A_j(x, \xi)$?

Make the substitutions $\xi_i \mapsto \partial_i$ and $\partial_i \mapsto x_i$:

$$A_j(x, \xi) \mapsto A_j(x, \partial) \in \mathbb{C}\langle x, \partial \rangle \quad \text{and} \quad p_k(\partial) \mapsto p_k(x) \in \mathbb{C}[x].$$

Fix one of the varieties V_j, then for all p_k we can prove that

$$A_j(x, \partial) \cdot p_k(q) = 0 \quad \text{for all } q \in V_j.$$

What is so special about these polynomials $A_j(x,\xi)$?

Make the substitutions $\xi_i \mapsto \partial_i$ and $\partial_i \mapsto x_i$:

$$A_j(x,\xi) \mapsto A_j(x,\partial) \in \mathbb{C}\langle x,\partial\rangle \quad \text{and} \quad p_k(\partial) \mapsto p_k(x) \in \mathbb{C}[x].$$

Fix one of the varieties V_j, then for all p_k we can prove that

$$A_j(x,\partial) \cdot p_k(q) = 0 \quad \text{for all } q \in V_j.$$

Equivalently, we have

$$A_j(x,\partial) \cdot p_k(x) \subset I(V_j) \subset \mathbb{C}[x].$$

By setting $I = (p_1(x), \ldots, p_m(x)) \subset \mathbb{C}[x]$, we obtain

$$A_j(x,\partial) \cdot I \subset I(V_j).$$

What is so special about these polynomials $A_j(x, \xi)$?

Make the substitutions $\xi_i \mapsto \partial_i$ and $\partial_i \mapsto x_i$:

$$A_j(x, \xi) \mapsto A_j(x, \partial) \in \mathbb{C}\langle x, \partial \rangle \quad \text{and} \quad p_k(\partial) \mapsto p_k(x) \in \mathbb{C}[x].$$

Fix one of the varieties V_j, then for all p_k we can prove that

$$A_j(x, \partial) \cdot p_k(q) = 0 \quad \text{for all } q \in V_j.$$

Equivalently, we have

$$A_j(x, \partial) \cdot p_k(x) \subset I(V_j) \subset \mathbb{C}[x].$$

By setting $I = (p_1(x), \ldots, p_m(x)) \subset \mathbb{C}[x]$, we obtain

$$A_j(x, \partial) \cdot I \subset I(V_j).$$

Noetherian operators

This is a theorem of "local nature". Suppose that there is only variety V_j, that is $V_1 = V_2 = \cdots = V_t$, then we are in the following situation:

- $I = (p_1(x), \ldots, p_m(x)) \subset \mathbb{C}[x]$ is a primary ideal and

$$A_j(x, \partial) \cdot I \subset \sqrt{I}.$$

Noetherian operators

This is a theorem of "local nature". Suppose that there is only variety V_j, that is $V_1 = V_2 = \cdots = V_t$, then we are in the following situation:

- $I = (p_1(x), \ldots, p_m(x)) \subset \mathbb{C}[x]$ is a primary ideal and

$$A_j(x, \partial) \cdot I \subset \sqrt{I}.$$

Condition of Ehrenpreis-Palamodov Theorem: Noetherian operators

$I \subset \mathbb{C}[x]$ primary ideal, if we find $\{A_j(x, \partial)\}_{j=1}^{t}$ such that

$$I = \left\{ f \in R \mid A_j(x, \partial) \cdot f \in \sqrt{I} \text{ for all } 1 \leq j \leq t \right\},$$

Noetherian operators

This is a theorem of "local nature". Suppose that there is only variety V_j, that is $V_1 = V_2 = \cdots = V_t$, then we are in the following situation:

- $I = (p_1(x), \ldots, p_m(x)) \subset \mathbb{C}[x]$ is a primary ideal and

$$A_j(x, \partial) \cdot I \subset \sqrt{I}.$$

Condition of Ehrenpreis-Palamodov Theorem: Noetherian operators

$I \subset \mathbb{C}[x]$ primary ideal, if we find $\{A_j(x, \partial)\}_{j=1}^{t}$ such that

$$I = \left\{ f \in R \mid A_j(x, \partial) \cdot f \in \sqrt{I} \text{ for all } 1 \leq j \leq t \right\},$$

then the statement of the theorem holds!

Our problem

Classical problem of finding Noetherian operators

Let $\mathfrak{p} \subset \mathbb{C}[x_1, \ldots, x_n]$ be a prime ideal and $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a $\mathfrak{p}$-primary ideal. Find differential operators $A_1, \ldots, A_t \in D_n(\mathbb{C}) = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$, such that

$$I = \{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \le j \le t \}.$$

Our problem

Classical problem of finding Noetherian operators

Let $\mathfrak{p} \subset \mathbb{C}[x_1, \ldots, x_n]$ be a prime ideal and $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a $\mathfrak{p}$-primary ideal. Find differential operators $A_1, \ldots, A_t \in D_n(\mathbb{C}) = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$, such that

$$I = \{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \leq j \leq t \}.$$

More general problem

Let R be a **Noetherian ring**, $\mathfrak{p} \subset R$ be a prime ideal and $I \subset R$ be a primary ideal. Describe I via "differential operators".

Some "algebraic" results

- The existence of Noetherian operators for **prime ideals** is due to Gröbner in the 1930's.

Some “algebraic” results

- The existence of Noetherian operators for **prime ideals** is due to Gröbner in the 1930’s.
- G. Brumfiel (1978). “Differential operators and primary ideals”. In: *J. Algebra* 51.2, pp. 375–398.

Some "algebraic" results

- The existence of Noetherian operators for **prime ideals** is due to Gröbner in the 1930's.
- G. Brumfiel (1978). "Differential operators and primary ideals". In: *J. Algebra* 51.2, pp. 375–398.
- Ulrich Oberst (1999). "The construction of Noetherian operators". In: *J. Algebra* 222.2, pp. 595–620.

Definition

Let R be a commutative ring and A be a subring. Let M, N be R-modules. The n-th order A-linear differential operators $\mathrm{Diff}^n_{R/A}(M, N) \subseteq \mathrm{Hom}_A(M, N)$ are defined inductively by:

1. $\mathrm{Diff}^0_{R/A}(M, N) := \mathrm{Hom}_R(M, N)$.
2. $\mathrm{Diff}^n_{R/A}(M, N) := \left\{\delta \in \mathrm{Hom}_A(M, N) \mid [\delta, r] \in \mathrm{Diff}^{n-1}_{R/A}(M, N) \text{ for all } r \in R\right\}$, where $[\delta, r](m) = \delta(rm) - r\delta(m)$ for all $m \in M$.

The A-linear differential operators are given by

$$\mathrm{Diff}_{R/A}(M, N) := \bigcup_{n=0}^{\infty} \mathrm{Diff}^n_{R/A}(M, N).$$

Definition

Let R be a commutative ring and A be a subring. Let M, N be R-modules. The n-th order A-linear differential operators $\mathrm{Diff}^n_{R/A}(M, N) \subseteq \mathrm{Hom}_A(M, N)$ are defined inductively by:

1. $\mathrm{Diff}^0_{R/A}(M, N) := \mathrm{Hom}_R(M, N)$.
2. $\mathrm{Diff}^n_{R/A}(M, N) := \left\{\delta \in \mathrm{Hom}_A(M, N) \mid [\delta, r] \in \mathrm{Diff}^{n-1}_{R/A}(M, N) \text{ for all } r \in R\right\}$, where $[\delta, r](m) = \delta(rm) - r\delta(m)$ for all $m \in M$.

The A-linear differential operators are given by

$$\mathrm{Diff}_{R/A}(M, N) := \bigcup_{n=0}^{\infty} \mathrm{Diff}^n_{R/A}(M, N).$$

Example

For $R = \mathbb{C}[x_1, \ldots, x_n]$ we have that

$$D_n(\mathbb{C}) = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle = \mathrm{Diff}_{R/\mathbb{C}}(R, R).$$

What “kind of differential operators” can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, and Gelfand 1972 showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

What “kind of differential operators” can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, and Gelfand 1972 showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator “∂_x” does not exist in $\mathrm{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, and Gelfand 1972 showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R,R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x,y,z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R,R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\text{Diff}_{R/\mathbb{C}}(R,R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3+y^3+z^3) = 3x^2$ (a contradiction).

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, and Gelfand 1972 showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x,y,z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3+y^3+z^3) = 3x^2$ (a contradiction).

In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

If $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$ were Noetherian operators from $\mathfrak{m}^2$ we would have

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \le i \le m \right\}$$

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, and Gelfand 1972 showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R,R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x,y,z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R,R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\text{Diff}_{R/\mathbb{C}}(R,R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3+y^3+z^3) = 3x^2$ (a contradiction).

In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

If $\delta_1,\ldots,\delta_m \in \text{Diff}_{R/\mathbb{C}}(R,R)$ were Noetherian operators from $\mathfrak{m}^2$ we would have

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \le i \le m \right\} \supseteq \mathfrak{m}$$

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, and Gelfand 1972 showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x,y,z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3+y^3+z^3) = 3x^2$ (a contradiction).

In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

If $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$ were Noetherian operators from $\mathfrak{m}^2$ we would have

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \le i \le m \right\} \supseteq \mathfrak{m}$$

(a contradiction).

(Simplified) Data

Let $\mathbb{k}$ be a field, R be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal.

(Simplified) Data

Let $\mathbb{k}$ be a field, R be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal.

The canonical map $\pi : R \twoheadrightarrow R/\mathfrak{p}$ induces a map

$$\operatorname{Diff}_{R/\mathbb{k}}(\pi) : \operatorname{Diff}_{R/\mathbb{k}}(R, R) \to \operatorname{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta.$$

(Simplified) Data

Let $\mathbb{k}$ be a field, R be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal.

The canonical map $\pi : R \twoheadrightarrow R/\mathfrak{p}$ induces a map

$$\mathrm{Diff}_{R/\mathbb{k}}(\pi) : \mathrm{Diff}_{R/\mathbb{k}}(R, R) \to \mathrm{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta.$$

If $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{k}}(R, R)$ and $I = \left\{ f \in R \mid \delta_i(f) \in \mathfrak{p} \text{ for all } 1 \leq i \leq m \right\}$ then

$$I = \left\{ f \in R \mid \overline{\delta_i}(f) = 0 \text{ for all } 1 \leq i \leq m \right\}.$$

(Simplified) Data

Let $\Bbbk$ be a field, R be a $\Bbbk$-algebra of finite type, $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal.

The canonical map $\pi : R \twoheadrightarrow R/\mathfrak{p}$ induces a map

$$\operatorname{Diff}_{R/\Bbbk}(\pi) : \operatorname{Diff}_{R/\Bbbk}(R, R) \to \operatorname{Diff}_{R/\Bbbk}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta.$$

If $\delta_1, \ldots, \delta_m \in \operatorname{Diff}_{R/\Bbbk}(R, R)$ and $I = \left\{ f \in R \mid \delta_i(f) \in \mathfrak{p} \text{ for all } 1 \leq i \leq m \right\}$ then

$$I = \left\{ f \in R \mid \overline{\delta_i}(f) = 0 \text{ for all } 1 \leq i \leq m \right\}.$$

Set $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$ and $\mathfrak{m} = (x, y, z)$. In $\operatorname{Diff}_{R/\mathbb{C}}(R, R/\mathfrak{m})$ we do have the operator "$\overline{\partial_x}$" because

$$0 = \overline{\partial_x}(0) = \overline{\partial_x}(x^3 + y^3 + z^3) = \overline{3x^2} = 0.$$

(Simplified) Data

Let $\Bbbk$ be a field, R be a $\Bbbk$-algebra of finite type, $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal.

The canonical map $\pi : R \twoheadrightarrow R/\mathfrak{p}$ induces a map

$$\mathrm{Diff}_{R/\Bbbk}(\pi) : \mathrm{Diff}_{R/\Bbbk}(R, R) \to \mathrm{Diff}_{R/\Bbbk}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta.$$

If $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\Bbbk}(R, R)$ and $I = \left\{ f \in R \mid \delta_i(f) \in \mathfrak{p} \text{ for all } 1 \leq i \leq m \right\}$ then

$$I = \left\{ f \in R \mid \overline{\delta_i}(f) = 0 \text{ for all } 1 \leq i \leq m \right\}.$$

Set $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$ and $\mathfrak{m} = (x, y, z)$. In $\mathrm{Diff}_{R/\mathbb{C}}(R, R/\mathfrak{m})$ we do have the operator "$\overline{\partial_x}$" because

$$0 = \overline{\partial_x}(0) = \overline{\partial_x}(x^3 + y^3 + z^3) = \overline{3x^2} = 0.$$

Therefore, $\mathrm{Diff}_{R/\Bbbk}(\pi)$ is **not surjective**.

The main result

Theorem

Let $\mathbb{k}$ *be a field,* R *be a* $\mathbb{k}$*-algebra of finite type,* $\mathfrak{p} \in \operatorname{Spec}(R)$ *be a prime ideal, and* $I \subset R$ *be a* $\mathfrak{p}$*-primary ideal. Then:*

1. *There exist* $\delta_1, \ldots, \delta_m \in \operatorname{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p})$ *such that*

$$I = \{f \in R \mid \delta_i(f) = 0 \ \textit{ for all } 1 \leq i \leq m\}.$$

The main result

Theorem

Let $\Bbbk$ *be a field,* R *be a* $\Bbbk$*-algebra of finite type,* $\mathfrak{p} \in \operatorname{Spec}(R)$ *be a prime ideal, and* $I \subset R$ *be a* $\mathfrak{p}$*-primary ideal. Then:*

1. *There exist* $\delta_1, \ldots, \delta_m \in \operatorname{Diff}_{R/\Bbbk}(R, R/\mathfrak{p})$ *such that*

$$I = \{f \in R \mid \delta_i(f) = 0 \text{ for all } 1 \leq i \leq m\}.$$

2. *If* $\Bbbk$ *is a perfect field and* $I \supseteq \mathfrak{p}^{n+1}$*, then there exist* $\delta_1, \ldots, \delta_m \in \operatorname{Diff}^n_{R/\Bbbk}(R, R/\mathfrak{p})$ *such that*

$$I = \{f \in R \mid \delta_i(f) = 0 \text{ for all } 1 \leq i \leq m\}.$$

Important result (from Grothendieck 1967 EGA IV)

If R is smooth over $\mathbb{k}$, then the map

$$\operatorname{Diff}_{R/\mathbb{k}}(\pi) : \operatorname{Diff}_{R/\mathbb{k}}(R, R) \to \operatorname{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta$$

is **surjective**.

Important result (from Grothendieck 1967 EGA IV)

If R is smooth over $\mathbb{k}$, then the map

$$\mathrm{Diff}_{R/\mathbb{k}}(\pi) : \mathrm{Diff}_{R/\mathbb{k}}(R, R) \to \mathrm{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta$$

is **surjective**.

Corollary

Suppose that R is smooth over $\mathbb{k}$. Then:

1. There exist $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{k}}(R, R)$ such that

$$I = \{f \in R \mid \delta_i(f) \in \mathfrak{p} \text{ for all } 1 \le i \le m\}.$$

Important result (from Grothendieck 1967 EGA IV)

If R is smooth over $\mathbb{k}$, then the map

$$\mathrm{Diff}_{R/\mathbb{k}}(\pi) : \mathrm{Diff}_{R/\mathbb{k}}(R, R) \to \mathrm{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta$$

is **surjective**.

Corollary

Suppose that R is smooth over $\mathbb{k}$. Then:

1. There exist $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{k}}(R, R)$ such that

$$I = \{f \in R \mid \delta_i(f) \in \mathfrak{p} \ \text{ for all } 1 \le i \le m\}.$$

2. If $\mathbb{k}$ is a perfect field and $I \supseteq \mathfrak{p}^{n+1}$, then there exist $\delta_1, \ldots, \delta_m \in \mathrm{Diff}^n_{R/\mathbb{k}}(R, R)$ such that

$$I = \{f \in R \mid \delta_i(f) \in \mathfrak{p} \ \text{ for all } 1 \le i \le m\}.$$

Non reduced schemes

Basic Question

It is well known that the reduced scheme structure of a scheme is unique. But, how to differentiate schemes with the same underlying topological space?

Non reduced schemes

Basic Question

It is well known that the reduced scheme structure of a scheme is unique. But, how to differentiate schemes with the same underlying topological space?

One approach could be to use differential operators.

Non reduced schemes

Basic Question

It is well known that the reduced scheme structure of a scheme is unique. But, how to differentiate schemes with the same underlying topological space?

One approach could be to use differential operators.

Example

- $R = \mathbb{C}[x]$ and $X = \mathbb{A}^1_{\mathbb{C}} = \text{Spec}(R)$.
- $X_n = \text{Spec}\left(R/(x^n)\right)$ – "n-th multiple point at 0".

Equivalently, we could give the following data:

$$X_n \quad \cong \quad \left(X_1 = \text{Spec}\left(R/(x)\right), \left\{1, \partial_x, \ldots, \partial_x^{n-1}\right\}\right).$$

Non reduced schemes

Basic Question

It is well known that the reduced scheme structure of a scheme is unique. But, how to differentiate schemes with the same underlying topological space?

One approach could be to use differential operators.

Example

- $R = \mathbb{C}[x]$ and $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}(R)$.
- $X_n = \operatorname{Spec}(R/(x^n))$ – "n-th multiple point at 0".

Equivalently, we could give the following data:

$$X_n \quad \cong \quad \left(X_1 = \operatorname{Spec}(R/(x)), \left\{1, \partial_x, \ldots, \partial_x^{n-1}\right\}\right).$$

Follows from: $(x^n) = \{f \in R \mid \partial_x^j(f) \in (x) \text{ for all } 1 \le j \le n-1\}$.

Zariski-Nagata Theorem

Let V be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The n-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.

Zariski-Nagata Theorem

Let V be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The n-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.

Zariski-Nagata Theorem

For all $n \geq 1$ we have

- $\mathfrak{p}^{(n)} = \bigcap_{q \in V} {\mathfrak{m}_q}^n$. ($\mathfrak{m}_q$ is the maximal ideal corresponding with q)

Zariski-Nagata Theorem

Let V be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The n-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.

Zariski-Nagata Theorem

For all $n \geq 1$ we have

- $\mathfrak{p}^{(n)} = \bigcap_{q \in V} \mathfrak{m}_q{}^n$. ($\mathfrak{m}_q$ is the maximal ideal corresponding with q)
- $\mathfrak{p}^{(n)} = \{f \in R \mid \frac{\partial^\alpha f}{\partial x^\alpha}(q) = 0 \quad \text{for all } |\alpha| \leq n-1 \text{ and } q \in V\}$.

Zariski-Nagata Theorem

Let V be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The n-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R$.

Zariski-Nagata Theorem

For all $n \geq 1$ we have

- $\mathfrak{p}^{(n)} = \bigcap_{q \in V} \mathfrak{m}_q{}^n$. ($\mathfrak{m}_q$ is the maximal ideal corresponding with q)
- $\mathfrak{p}^{(n)} = \{f \in R \mid \frac{\partial^\alpha f}{\partial x^\alpha}(q) = 0 \quad \text{for all } |\alpha| \leq n-1 \text{ and } q \in V\}$.

Zariski-Nagata Theorem

Let V be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The n-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R$.

Zariski-Nagata Theorem

For all $n \geq 1$ we have

- $\mathfrak{p}^{(n)} = \bigcap_{q \in V} {\mathfrak{m}_q}^n$. ($\mathfrak{m}_q$ is the maximal ideal corresponding with q)
- $\mathfrak{p}^{(n)} = \{f \in R \mid \frac{\partial^\alpha f}{\partial x^\alpha}(q) = 0 \quad \text{for all } |\alpha| \leq n-1 \text{ and } q \in V\}$.

Easy reinterpretation (in the sense of Noetherian operators)

$\mathfrak{p}^{(n)} = \{f \in R \mid \frac{\partial^\alpha}{\partial x^\alpha}(f) \in \mathfrak{p} \quad \text{for all } |\alpha| \leq n-1\}$

Extension of Zariski-Nagata Theorem

Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)

Let $\Bbbk$ be a perfect field, R be a **smooth** algebra of finite type over $\Bbbk$, and $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal. Set

$$\mathfrak{p}^{\langle n \rangle} := \left\{ f \in R \mid \delta(f) \in \mathfrak{p} \text{ for all } \delta \in \operatorname{Diff}^{n-1}_{R/\Bbbk}(R, R) \right\}.$$

Then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\langle n \rangle}$.

Extension of Zariski-Nagata Theorem

Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)

Let $\Bbbk$ be a perfect field, R be a **smooth** algebra of finite type over $\Bbbk$, and $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal. Set

$$\mathfrak{p}^{\langle n \rangle} := \left\{ f \in R \mid \delta(f) \in \mathfrak{p} \text{ for all } \delta \in \operatorname{Diff}^{n-1}_{R/\Bbbk}(R,R) \right\}.$$

Then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\langle n \rangle}$.

We can proof the following further extension:

Theorem

$\Bbbk$ perfect field, R algebra of finite type over $\Bbbk$, and $\mathfrak{p} \in \operatorname{Spec}(R)$. Set

$$\mathfrak{p}^{\{n\}} := \left\{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in \operatorname{Diff}^{n-1}_{R/\Bbbk}(R, R/\mathfrak{p}) \right\}.$$

Then:

- $\mathfrak{p}^{(n)} = \mathfrak{p}^{\{n\}}$.

Extension of Zariski-Nagata Theorem

Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)

Let $\Bbbk$ be a perfect field, R be a **smooth** algebra of finite type over $\Bbbk$, and $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal. Set

$$\mathfrak{p}^{\langle n \rangle} := \left\{ f \in R \mid \delta(f) \in \mathfrak{p} \text{ for all } \delta \in \operatorname{Diff}^{n-1}_{R/\Bbbk}(R, R) \right\}.$$

Then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\langle n \rangle}$.

We can proof the following further extension:

Theorem

$\Bbbk$ perfect field, R algebra of finite type over $\Bbbk$, and $\mathfrak{p} \in \operatorname{Spec}(R)$. Set

$$\mathfrak{p}^{\{n\}} := \left\{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in \operatorname{Diff}^{n-1}_{R/\Bbbk}(R, R/\mathfrak{p}) \right\}.$$

Then:

- $\mathfrak{p}^{(n)} = \mathfrak{p}^{\{n\}}$.
- If R is **smooth** over $\Bbbk$, then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\{n\}} = \mathfrak{p}^{\langle n \rangle}$.

Lets compute **THE EXAMPLE**

For the rest of the presentation we fix:

- $R = \mathbb{C}[x_1, x_2, x_3]$.
- $I = (x_1 x_3 - x_2, x_2^2, x_3^2) \subset R$.
- $\mathfrak{p} = (x_2, x_3) = \sqrt{I} \subset R$.

The Noetherian operators for I are $A_1 = 1$ and $A_2 = \partial_3 + x_1 \partial_2$, that is

$$I = \left\{ f \in R \mid f \in \mathfrak{p} \;\text{ and }\; (\partial_2 + x_1 \partial_2) \cdot f \in \mathfrak{p} \right\}.$$

Lets compute **THE EXAMPLE**

For the rest of the presentation we fix:

- $R = \mathbb{C}[x_1, x_2, x_3]$.
- $I = (x_1x_3 - x_2, x_2^2, x_3^2) \subset R$.
- $\mathfrak{p} = (x_2, x_3) = \sqrt{I} \subset R$.

The Noetherian operators for I are $A_1 = 1$ and $A_2 = \partial_3 + x_1\partial_2$, that is

$$I = \left\{ f \in R \mid f \in \mathfrak{p} \;\text{ and }\; (\partial_2 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}.$$

This example was given by Palamodov as a counter-example to Ehrenpreis claim: "*Noetherian operators can always be chosen with constant coefficients*".

Probably the simplest example, because the claim holds for the monomial and zero-dimensional cases (see Sturmfels 2002).

Since $I \supset \mathfrak{p}^2$, our main theorem says that we can find differential operators inside $\mathrm{Diff}^1_{R/\mathbb{C}}(R, R)$.

Since $I \supset \mathfrak{p}^2$, our main theorem says that we can find differential operators inside $\mathrm{Diff}^1_{R/\mathbb{C}}(R, R)$.

$\mathrm{Diff}^1_{R/\mathbb{C}}(R, R) = \{\delta \in \mathrm{Hom}_{\mathbb{C}}(R, R) \mid [[\delta, r_1], r_2] = 0 \text{ for all } r_1, r_2 \in R\}$.
Now, $\mathrm{Hom}_{\mathbb{C}}(R, R)$ is a $(R - R)$-bimodule or a $(R \otimes_{\mathbb{C}} R)$-module:

$$(r_1 \otimes_{\mathbb{C}} r_2) \cdot \delta \text{ is defined by } ((r_1 \otimes_{\mathbb{C}} r_2) \cdot \delta)(t) = r_1 \delta(r_2 t).$$

Since $I \supset \mathfrak{p}^2$, our main theorem says that we can find differential operators inside $\mathrm{Diff}^1_{R/\mathbb{C}}(R, R)$.

$\mathrm{Diff}^1_{R/\mathbb{C}}(R, R) = \{\delta \in \mathrm{Hom}_{\mathbb{C}}(R, R) \mid [[\delta, r_1], r_2] = 0 \text{ for all } r_1, r_2 \in R\}$.
Now, $\mathrm{Hom}_{\mathbb{C}}(R, R)$ is a $(R - R)$-bimodule or a $(R \otimes_{\mathbb{C}} R)$-module:

$$(r_1 \otimes_{\mathbb{C}} r_2) \cdot \delta \text{ is defined by } ((r_1 \otimes_{\mathbb{C}} r_2) \cdot \delta)(t) = r_1 \delta(r_2 t).$$

Therefore $\mathrm{Diff}^1_{R/\mathbb{C}}(R, R)$

$= \{\delta \in \mathrm{Hom}_{\mathbb{C}}(R, R) \mid (1 \otimes_{\mathbb{C}} r_2 - r_2 \otimes_{\mathbb{C}} 1)(1 \otimes_{\mathbb{C}} r_1 - r_1 \otimes_{\mathbb{C}} 1) \cdot \delta = 0 \text{ for all } r_1, r_2 \in R\}$.

Since $I \supset \mathfrak{p}^2$, our main theorem says that we can find differential operators inside $\text{Diff}^1_{R/\mathbb{C}}(R, R)$.

$\text{Diff}^1_{R/\mathbb{C}}(R, R) = \{\delta \in \text{Hom}_{\mathbb{C}}(R, R) \mid [[\delta, r_1], r_2] = 0 \text{ for all } r_1, r_2 \in R\}$.
Now, $\text{Hom}_{\mathbb{C}}(R, R)$ is a $(R - R)$-bimodule or a $(R \otimes_{\mathbb{C}} R)$-module:

$$(r_1 \otimes_{\mathbb{C}} r_2) \cdot \delta \text{ is defined by } ((r_1 \otimes_{\mathbb{C}} r_2) \cdot \delta)(t) = r_1 \delta(r_2 t).$$

Therefore $\text{Diff}^1_{R/\mathbb{C}}(R, R)$

$= \{\delta \in \text{Hom}_{\mathbb{C}}(R, R) \mid (1 \otimes_{\mathbb{C}} r_2 - r_2 \otimes_{\mathbb{C}} 1)(1 \otimes_{\mathbb{C}} r_1 - r_1 \otimes_{\mathbb{C}} 1) \cdot \delta = 0 \text{ for all } r_1, r_2 \in R\}$.

By writing $\Delta_{R/\mathbb{C}} = (1 \otimes_{\mathbb{C}} x_1 - x_1 \otimes_{\mathbb{C}} 1,\ 1 \otimes_{\mathbb{C}} x_2 - x_2 \otimes_{\mathbb{C}} 1,\ 1 \otimes_{\mathbb{C}} x_3 - x_3 \otimes_{\mathbb{C}} 1)$:

$$\begin{aligned}\text{Diff}^1_{R/\mathbb{C}}(R, R) &= \{\delta \in \text{Hom}_{\mathbb{C}}(R, R) \mid \Delta^2_{R/\mathbb{C}} \cdot \delta = 0\} \\ &\cong \{\delta \in \text{Hom}_{\mathbb{C}}(R, \text{Hom}_R(R, R)) \mid \Delta^2_{R/\mathbb{C}} \cdot \delta = 0\} \\ &\cong \{\delta \in \text{Hom}_R(R \otimes_{\mathbb{C}} R, R) \mid \Delta^2_{R/\mathbb{C}} \cdot \delta = 0\}\end{aligned}$$

$$\text{Diff}^1_{R/\mathbb{C}}(R, R) \cong \text{Hom}_R\left(\frac{R \otimes_{\mathbb{C}} R}{\Delta^2_{R/\mathbb{C}}}, R\right).$$

Proposition (Grothendieck 1967, Heyneman and Sweedler 1969)

We have the isomorphism

$$\mathrm{Hom}_R\left(\frac{R \otimes_{\mathbb{C}} R}{\Delta_{R/\mathbb{C}}^{n+1}}, R\right) \xrightarrow{\cong} \mathrm{Diff}^n_{R/\mathbb{C}}(R, R)$$

$$\varphi \mapsto \varphi \circ d^n$$

where

$$d^n : R \to \frac{R \otimes_{\mathbb{C}} R}{\Delta_{R/\mathbb{C}}^{n+1}}, \qquad r \mapsto \overline{1 \otimes_{\mathbb{C}} r}.$$

Proposition (Grothendieck 1967, Heyneman and Sweedler 1969)

We have the isomorphism

$$\mathrm{Hom}_R\left(\frac{R\otimes_{\mathbb{C}} R}{\Delta_{R/\mathbb{C}}^{n+1}}, R\right) \xrightarrow{\cong} \mathrm{Diff}^n_{R/\mathbb{C}}(R,R)$$

$$\varphi \mapsto \varphi \circ d^n$$

where

$$d^n : R \to \frac{R\otimes_{\mathbb{C}} R}{\Delta_{R/\mathbb{C}}^{n+1}}, \qquad r \mapsto \overline{1\otimes_{\mathbb{C}} r}.$$

Reminder ("Derivations are the dual of Kähler differentials")

We have the isomorphism

$$\mathrm{Hom}_R\left(\Omega_{R/\mathbb{C}}, R\right) \xrightarrow{\cong} \mathrm{Der}_{\mathbb{C}}(R)$$

$$\varphi \mapsto \varphi \circ d$$

where

$$d : R \to \Omega_{R/\mathbb{C}} = \frac{\Delta_{R/\mathbb{C}}}{\Delta_{R/\mathbb{C}}^2}, \qquad r \mapsto \overline{1\otimes_{\mathbb{C}} r - r\otimes_{\mathbb{C}} 1}.$$

We have $R = \mathbb{C}[x_1, x_2, x_3]$, $I = (x_1x_3 - x_2, x_2^2, x_3^2)$, $\mathfrak{p} = (x_2, x_3)$. We want to show $I = \left\{ f \in R \mid f \in \mathfrak{p} \text{ and } (\partial_3 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}$.

We have $R = \mathbb{C}[x_1, x_2, x_3]$, $I = (x_1x_3 - x_2, x_2^2, x_3^2)$, $\mathfrak{p} = (x_2, x_3)$. We want to show $I = \left\{ f \in R \mid f \in \mathfrak{p} \text{ and } (\partial_3 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}$.

STEP 1. Since $\mathbb{C}[x_1] \cap \mathfrak{p} = 0$ $\left(V(\mathfrak{p}) = \{(\alpha, 0, 0) \mid \alpha \in \mathbb{C}\} \right)$ we will "naively" consider the ring $R' = \mathbb{C}(x_1)[x_2, x_3]$ and the differential operators inside

$$\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R', R') \subset D_2(\mathbb{C}(x_1)) = \mathbb{C}(x_1)\langle x_2, x_3, \partial_2, \partial_3 \rangle.$$

We have $R = \mathbb{C}[x_1, x_2, x_3]$, $I = (x_1x_3 - x_2, x_2^2, x_3^2)$, $\mathfrak{p} = (x_2, x_3)$. We want to show $I = \left\{ f \in R \mid f \in \mathfrak{p} \text{ and } (\partial_3 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}$.

STEP 1. Since $\mathbb{C}[x_1] \cap \mathfrak{p} = 0$ $\Big(V(\mathfrak{p}) = \{(\alpha, 0, 0) \mid \alpha \in \mathbb{C}\}\Big)$ we will "naively" consider the ring $R' = \mathbb{C}(x_1)[x_2, x_3]$ and the differential operators inside

$$\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R', R') \subset D_2(\mathbb{C}(x_1)) = \mathbb{C}(x_1)\langle x_2, x_3, \partial_2, \partial_3 \rangle.$$

STEP 2. The canonical map $R' \twoheadrightarrow R'/(x_2, x_3) \cong \mathbb{C}(x_1)$ induces the map

$$\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R', R') \to \mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R', \mathbb{C}(x_1)).$$

We have $R=\mathbb{C}[x_1,x_2,x_3]$, $I=(x_1x_3-x_2,x_2^2,x_3^2)$, $\mathfrak{p}=(x_2,x_3)$. We want to show $I=\left\{f\in R \mid f\in\mathfrak{p} \text{ and } (\partial_3+x_1\partial_2)\cdot f\in\mathfrak{p}\right\}$.

STEP 1. Since $\mathbb{C}[x_1]\cap\mathfrak{p}=0$ $\Big(V(\mathfrak{p})=\{(\alpha,0,0)\mid\alpha\in\mathbb{C}\}\Big)$ we will "naively" consider the ring $R'=\mathbb{C}(x_1)[x_2,x_3]$ and the differential operators inside

$$\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R',R') \;\subset\; D_2\left(\mathbb{C}(x_1)\right)=\mathbb{C}(x_1)\langle x_2,x_3,\partial_2,\partial_3\rangle.$$

STEP 2. The canonical map $R'\twoheadrightarrow R'/(x_2,x_3)\cong\mathbb{C}(x_1)$ induces the map

$$\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R',R')\to\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R',\mathbb{C}(x_1)).$$

STEP 3. $$\begin{aligned}\mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R',\mathbb{C}(x_1)) &\cong \mathrm{Hom}_{R'}\left(\frac{R'\otimes R'}{(1\otimes x_2-x_2\otimes 1,1\otimes x_3-x_3\otimes 1)^2},\mathbb{C}(x_1)\right)\\ &\cong \mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{R'\otimes R'}{\left(x_2\otimes 1,\ x_3\otimes 1,\ (1\otimes x_2-x_2\otimes 1,1\otimes x_3-x_3\otimes 1)^2\right)},\mathbb{C}(x_1)\right)\\ &\cong \mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{1\otimes R'}{(1\otimes x_2,1\otimes x_3)^2},\mathbb{C}(x_1)\right).\end{aligned}$$

We have $R = \mathbb{C}[x_1, x_2, x_3]$, $I = (x_1x_3 - x_2, x_2^2, x_3^2)$, $\mathfrak{p} = (x_2, x_3)$. We want to show $I = \left\{ f \in R \mid f \in \mathfrak{p} \;\text{ and }\; (\partial_3 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}$.

We have $R = \mathbb{C}[x_1, x_2, x_3]$, $I = (x_1x_3 - x_2, x_2^2, x_3^2)$, $\mathfrak{p} = (x_2, x_3)$. We want to show $I = \left\{ f \in R \mid f \in \mathfrak{p} \text{ and } (\partial_3 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}$.

STEP 4. Since $(x_2, x_3)^2 \subset I' = (x_1x_3 - x_2, x_2^2, x_3^2) \subset R'$, then we have the inclusion

$$\mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{1\otimes R'}{1\otimes I'}, \mathbb{C}(x_1)\right) \subset \mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{1\otimes R'}{(1\otimes x_2, 1\otimes x_3)^2}, \mathbb{C}(x_1)\right) \cong \mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R', \mathbb{C}(x_1)).$$

We have $R = \mathbb{C}[x_1, x_2, x_3]$, $I = (x_1x_3 - x_2, x_2^2, x_3^2)$, $\mathfrak{p} = (x_2, x_3)$. We want to show $I = \left\{ f \in R \mid f \in \mathfrak{p} \text{ and } (\partial_3 + x_1\partial_2) \cdot f \in \mathfrak{p} \right\}$.

STEP 4. Since $(x_2, x_3)^2 \subset I' = (x_1x_3 - x_2, x_2^2, x_3^2) \subset R'$, then we have the inclusion

$$\mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{1\otimes R'}{1\otimes I'}, \mathbb{C}(x_1)\right) \subset \mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{1\otimes R'}{(1\otimes x_2, 1\otimes x_3)^2}, \mathbb{C}(x_1)\right) \cong \mathrm{Diff}^1_{R'/\mathbb{C}(x_1)}(R', \mathbb{C}(x_1)).$$

STEP 5. Note that $1 \otimes_{\mathbb{C}(x_1)} R' \cong R'$, that $1 \otimes_{\mathbb{C}(x_1)} I' \cong I'$ and that

$$I' = \left\{ f \in R' \mid (\varphi \circ d^1)(f) = \varphi(1 \otimes f) = 0 \text{ for all } \varphi \in \mathrm{Hom}_{\mathbb{C}(x_1)}\left(\frac{1\otimes R'}{1\otimes I'}, \mathbb{C}(x_1)\right) \right\}.$$

STEP 6. We have that $\frac{1\otimes R'}{1\otimes I'} \cong \frac{\mathbb{C}(x_1)[x_2,x_3]}{(x_1x_3 - x_2, x_2^2, x_3^2)} \cong \mathbb{C}(x_1) \cdot \{\overline{1}, \overline{x_3}\}$ is a **dimension two** $\mathbb{C}(x_1)$-vector space.

By **pulling back** the dual basis $\mathbb{C}(x_1) \cdot \{(\overline{1})^*, (\overline{x_3})^*\}$ we obtain the two Noetherian operators $A_1 = 1$ and $A_2 = \partial_3 + x_1\partial_2$ inside

$$D_3(\mathbb{C}) = \mathbb{C}\langle x_1, x_2, x_3, \partial_1, \partial_2, \partial_3 \rangle.$$

References I

Bernstein, I. N., I. M. Gelfand, and S. I. Gelfand (1972). "Differential operators on a cubic cone". In: *Uspehi Mat. Nauk* 27.1(163), pp. 185–190. ISSN: 0042-1316.

Brumfiel, G. (1978). "Differential operators and primary ideals". In: *J. Algebra* 51.2, pp. 375–398.

Dao, Hailong, Alessandro De Stefani, Eloísa Grifo, Craig Huneke, and Luis Núñez-Betancourt (2018). "Symbolic powers of ideals". In: *Singularities and foliations. geometry, topology and applications*. Vol. 222. Springer Proc. Math. Stat. Springer, Cham, pp. 387–432.

Grothendieck, Alexander (1967). "Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie". fr. In: *Publications Mathématiques de l'IHÉS* 32, pp. 5–361.

Heyneman, Robert G. and Moss Eisenberg Sweedler (1969). "Affine Hopf algebras. I". In: *J. Algebra* 13, pp. 192–241.

References II

Oberst, Ulrich (1999). "The construction of Noetherian operators". In: *J. Algebra* 222.2, pp. 595–620.

Sturmfels, Bernd (2002). *Solving systems of polynomial equations*. Vol. 97. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, pp. viii+152. ISBN: 0-8218-3251-4.

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Thanks a lot!

Danke vielmals!