Empty simplices of large width

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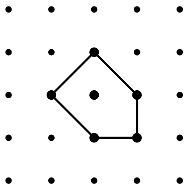

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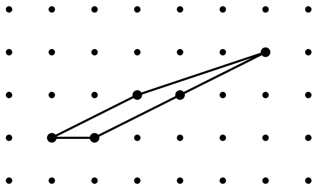

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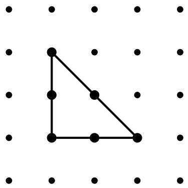

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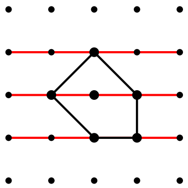

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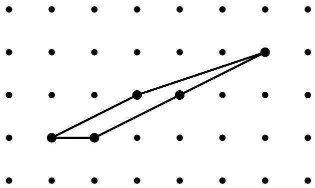

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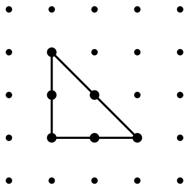

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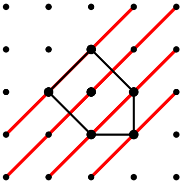

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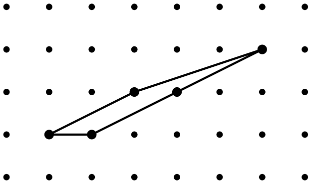

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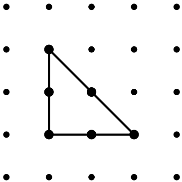

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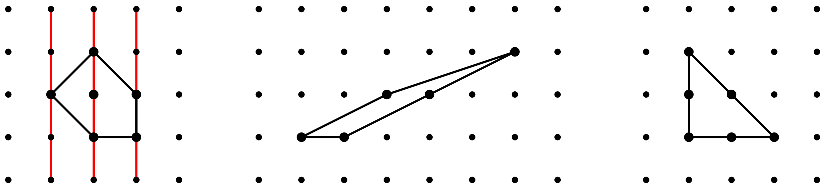

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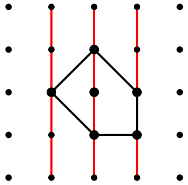

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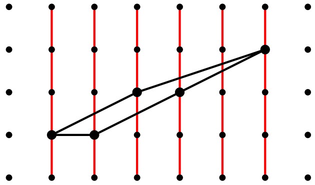

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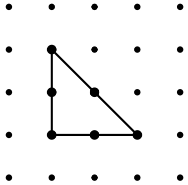

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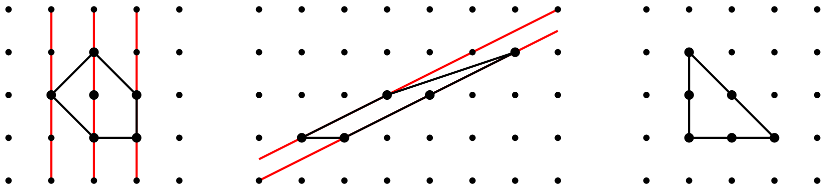

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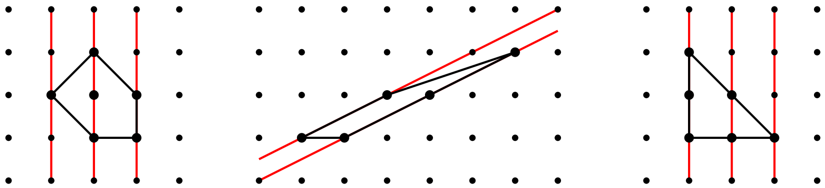

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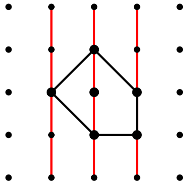

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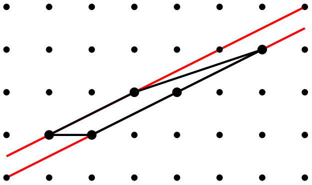

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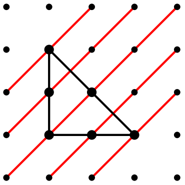

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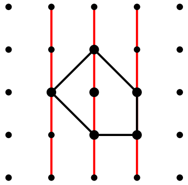

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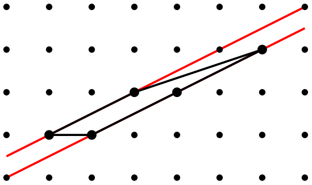

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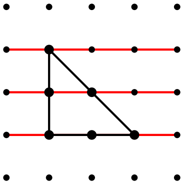

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Hollow convex bodies are flat

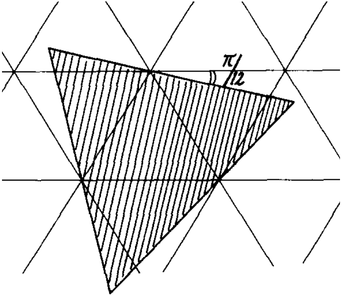
K is lattice-free or hollow if $\text{int}(K) \cap \Lambda = \emptyset$

Theorem (Flatness Theorem)

For each dimension d,

$$\text{Flt}_d := \sup_{K \text{ hollow}} \text{width}_\Lambda(K) < \infty.$$

Known values: $\text{Flt}_1 = 1$; $\quad \text{Flt}_2 = 1 + 3/\sqrt{2} \simeq 2.1547$ (Hurkens 1990)



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Common guess: $\mathrm{Flt}_d \in \Theta(d)$ (perhaps modulo poly-log factors).

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The last remark implies:

Corollary (Codenotti-S.-2019)

$$\lim_{d\to\infty} \frac{\mathrm{Flt}_d}{d} \geq \frac{\mathrm{Flt}_3}{3} \geq \frac{2+\sqrt{2}}{3} \approx 1.1381\ldots$$

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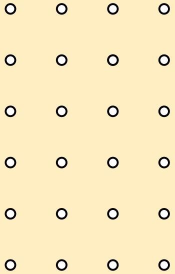

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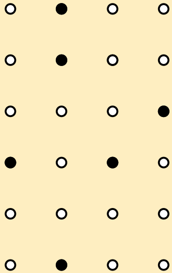

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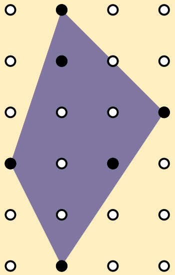

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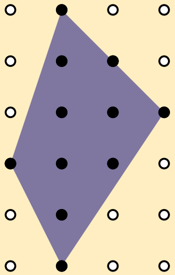

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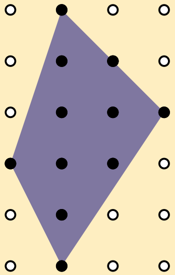

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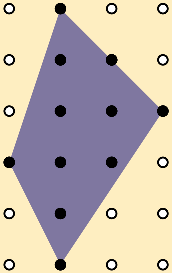

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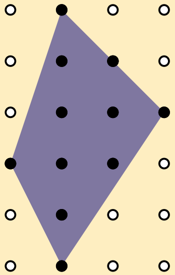

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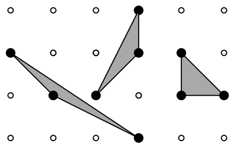
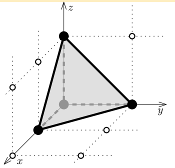
  E.g.: empty d-simplex $\Leftrightarrow$ lattice d-polytope with exacty $d+1$ lattice points.





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- They correspond to *terminal quotient singularities* in the minimal model program.
- Better understanding of hollow polytopes may lead to better bounds for the "flatness constant".

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Theorem (Classification of hollow polygons) The hollow polygons are the polygons of width one and the second dilation of a unimodular triangle.

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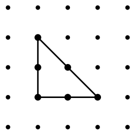
- **Dimension** 2: infinitely many *hollow* polygons (and triangles), but only one *empty* triangle, the unimodular one ($:\Leftrightarrow$ vertices are an affine basis for the lattice $\Leftrightarrow$ normalized volume $= 1$).

Corollary (Pick's theorem): If P is a lattice polygon with b and i lattice points in its boundary and interior, then $\text{area}(P) = (b + 2i - 2)$.

Theorem (Classification of hollow polygons) The hollow polygons are the polygons of width one and the second dilation of a unimodular triangle.



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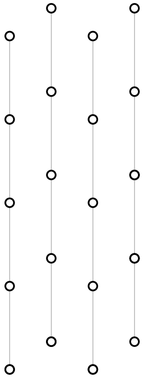

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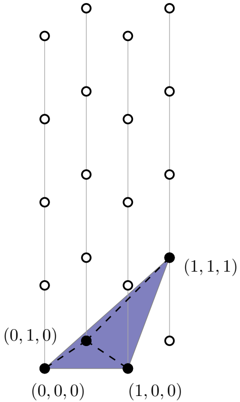

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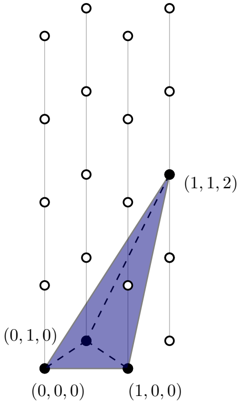

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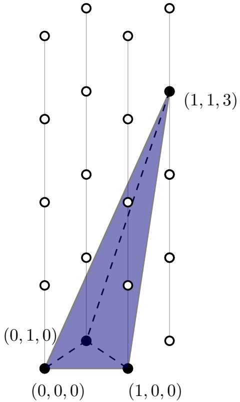

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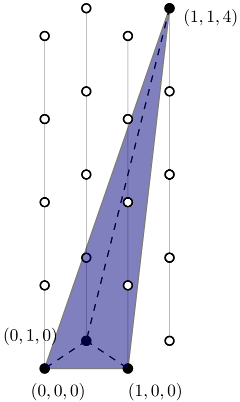

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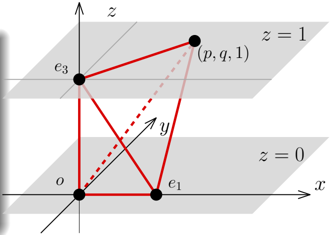
Theorem (White 1964)

Every empty tetrahedron has **width one**. *Hence it is equivalent to* $\Delta(p,q) :=$ $\operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$, *for some* $q \in \mathbb{N}$, $p \in \mathbb{Z}$, $\gcd(p,q) = 1$.



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- Among the empty 4-simplices of (normalized) volume up to 1000 those of width larger than two have volume ≤ 179. (There are 178 of width three plus one of width 4 and volume 101).

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Observe that $|\Lambda/\Lambda_\Delta|$ equals the normalized volume ($=$ the *determinant*) of Δ.

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There are exactly 2461 empty 4-simplices that *do not* project. Their volumes range from 24 to 419. There is **one of width** 4 (volume=101), 178 of width three (volumes $\in [49, 179]$), and the rest have width two.

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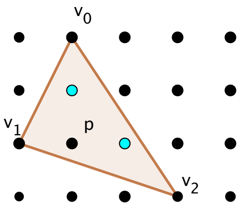
Q: Is there anything special about the one of width four?

A. YES; it happens to have an affine symmetry.

Barycentric coordinates

If $\Delta = (v_0, \ldots, v_d) \subset \mathbb{R}^d$ is a simplex, every point $p \in \mathbb{R}^d$ can be expressed uniquely as $\sum_{i=0}^{d} \beta_i v_i$ with $\sum_i \beta_i = 1$. The vector $(\beta_0, \ldots, \beta_d)$ are the *barycentric coordinates of p w.r.t. Δ*.



$$p = \frac{1}{7} v_0 + \frac{2}{7} v_1 + \frac{4}{7} v_2$$

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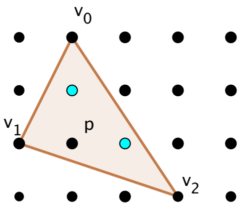

If Δ is a lattice simplex of (normalized) volume N and $p \in \Lambda$, then $\beta_i \in \frac{1}{N}\mathbb{Z}, \quad \forall i$. Hence, we write

$$(\beta_0, \ldots, \beta_d) = \frac{1}{N}(b_0, \ldots, b_d)$$

with $(b_0, \ldots, b_d) \in \mathbb{Z}^{d+1}$ and $\sum_i b_i = N$.

Cyclic simplices

Suppose now that Δ is a *cyclic lattice simplex*. That is, the quotient $\Lambda/\Lambda_\Delta \cong \mathbb{Z}_N$, where Λ_Δ is the lattice generated by the vertices of Δ. Then, Δ is completely characterized by its volume N and the vector $(b_0, \ldots, b_d) \in (\mathbb{Z}_N)^{d+1}$ representing the (normalized) barycentric coordinates of a generator of Λ/Λ_Δ.

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We are only interested in b_i only $\mod N$ because different representatives of the same $\tilde{p} \in \Lambda/\Lambda_\Delta$ have barycentric coordinates differing by an integer vector.

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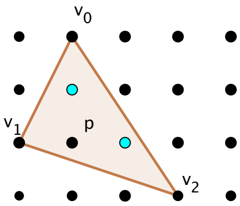

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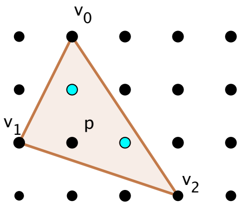

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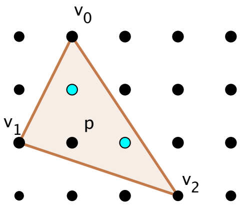

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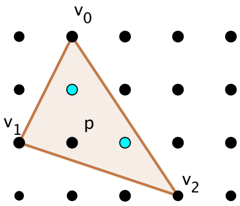

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Cyclic simplices with cyclic symmetry

Lemma

Let Δ be a cyclic simplex of prime volume N, defined by a $(d+1)$-tuple $(b_0, \dots, b_d)$. If there is an automorphism of Δ permuting cyclically its vertices, then $b_0, \dots, b_d$ are $(d+1)$-th roots of unity modulo N.

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Proof.

$(b_0, \dots, b_d)$ and $(b_1, \dots, b_d, b_0)$ must be multiples of one another in $(\mathbb{Z}_N)^{d+1}$, since they generate the same subgroup. If $k \in \mathbb{Z}_N$ is the scaling factor, then Hence,

$$b_i = b_0 k^i, \forall i \quad \text{and} \quad k^{d+1} \equiv 1 \pmod{N}.$$

Multiplying by b_0^{-1}, we get $(b_0, \dots, b_d) = (1, k, \dots, k^d)$. $\square$

Cyclic simplices with cyclic symmetry

Definition

Let $N \in \mathbb{N}$ be a prime and let $d < N$ be such that $d + 1$ divides $N - 1$. Let k be a $(d + 1)$-th roots of unity modulo N. The cyclotomic $\mathrm{Cycl}(N, d)$ is the cyclic d-simplex of volume N with generator $(1, k, \ldots, k^d)$.

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- The smallest empty 4-simplex of width 3 has volume 41 and generator $(1, -4, 16, -64, 256) = (1, -4, 16, -23, 10)$.

Exploring cyclotomic simplices

We have computed **all** the cyclotomic simplices with $d+1 \in \{5, 7, 11\}$ and $N \leq 2^{31}$, checked their emptyness and computed their width. There are:

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$d = 4$: exactly 4 empty cyclotomic simplices:

N	$(b_0, \ldots, b_4)$	width
11	(1, 4, 5, 9, 3)	2
41	(1, 10, 18, 16, 37)	3
61	(1, 9, 20, 58, 34)	3
101	(1, 95, 36, 87, 84)	4

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$d=6$: exactly 88 empty cyclotomic simplices, with volumes $\leq 17\,683$; six of them of width six, with volumes 6 301, 10 753, 11 117, 15 121, 16 493, and 17 683.

Exploring cyclotomic simplices

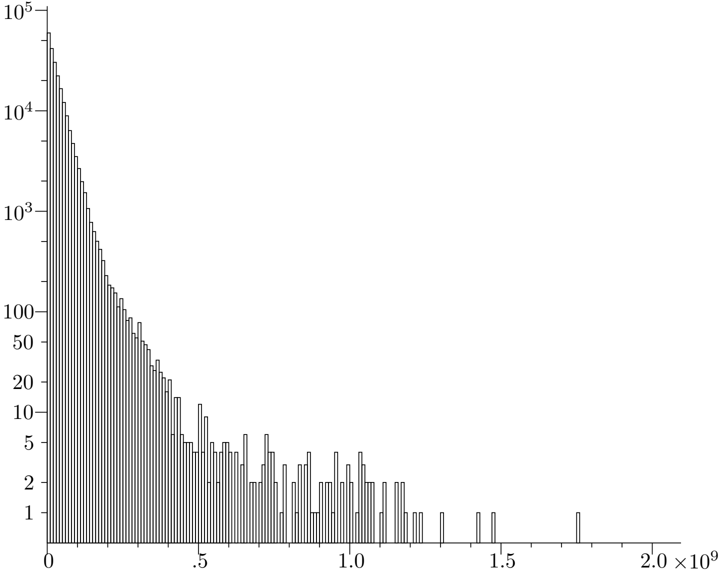
$d = 10$: there are 218 075 empty cyclotomic simplices (of volume $\leq 2^{31}$). Their maximum width is 11, attained by the following five:

$$\mathrm{Cycl}(10,\ 582\,595\,883),\quad \mathrm{Cycl}(10,\ 728\,807\,201),$$
$$\mathrm{Cycl}(10,\ 976\,965\,023),\quad \mathrm{Cycl}(10,\ 1\,066\,142\,419),$$
$$\mathrm{Cycl}(10,\ 1\,113\,718\,783).$$

Remark

For $d = 4, 6$ it is clear that our lists are complete. For $d = 10$ the list should also be. Between 10^9 and $2^{31} \approx 2{\cdot}10^9$ there are only 30, and only one above $1.5{\cdot}10^9$, of volume 1 757 211 061 and width 10. [a]

[a] Breaking news; according to one of my coauthors we have now extended our computations to $N \leq 2^{33}$ and no new empty cyclotomic 10-simplices arise.



Number of empty cyclotomic 10-simplices vs. volume

Symmetric simplices via circulant matrices

A more direct way to impose cyclic symmetry on lattice simplices is to take as vertices the columns of a circulant matrix.

Let $v = (v_0, \ldots, v_d) \in \mathbb{Z}^{d+1}$. For $i \in [d]$, let $v^{(i)}$ be the vector obtained from v by a cyclic shift of i places. That is:

$$v^{(i)} := (v_{d+1-i}, \ldots, v_d, v_0, \ldots, v_{d-i})$$

Definition

If the $v^{(i)}$'s defined above are linearly independent, we call

$$\mathrm{conv}(v^{(0)}, \ldots, v^{(d)})$$

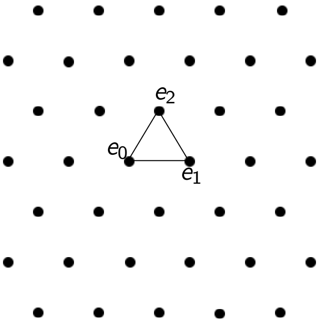
the circulant simplex (of dimension d) with generator v.

A particular circulant matrix

We define Circ$(d, m,)$ as the *circulant simplex* defined by the matrix

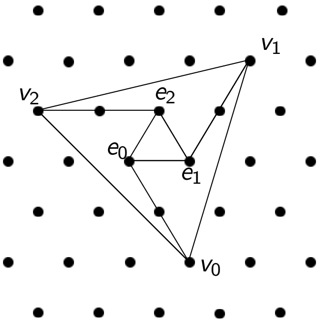
$$M(d,m) := \begin{pmatrix} 1 & -m & 0 & \dots & \dots & 0 & m \\ m & 1 & -m & \ddots & & & 0 \\ 0 & m & 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & -m & 0 \\ 0 & & & \ddots & m & 1 & -m \\ -m & 0 & \dots & \dots & 0 & m & 1 \end{pmatrix}$$

A particular circulant matrix



The standard simplex

A particular circulant matrix



The simplex Circ(d, m) for $d = m = 2$. The i-th vertex v_i equals

$$v_i = e_i + ma_i,$$

where $a_i := e_{i+1} - e_{i-1}, \quad i = 0, \dots, d.$

A particular circulant matrix

Lemma

For every even $d \geq 2$, and every $m \in \mathbb{R}$ we have

$$\mathrm{Vol}(\mathrm{Circ}(d,m)) = \det(M(d,m)) = \sum_{i=0}^{d/2} \frac{d+1}{d+1-i}\binom{d+1-i}{i} m^{2i}$$
$$= 1 + \sum_{i=1}^{d/2} \frac{d+1}{i}\binom{d-i}{i-1} m^{2i}.$$

Example

$\mathrm{Vol}(\mathrm{Circ}(4,2)) = 1 + \frac{5}{1}\binom{3}{0}2^2 + \frac{5}{2}\binom{2}{1}2^4 = 1 + 20 + 80 = 101$

$\mathrm{Vol}(\mathrm{Circ}(6,3)) = 1 + \frac{7}{1}\binom{5}{0}3^2 + \frac{7}{2}\binom{4}{1}3^4 + \frac{7}{3}\binom{3}{2}3^6 = \cdots = 6301$

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These volumes coincide with those of the unique widest empty 4-simplex ($\mathrm{Cycl}(101,4)$) and the smallest among the six cyclotomic empty 6-simplices of width 6 ($\mathrm{Cycl}(6301,6)$).

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This led us to conjecture that *for every even d, the simplex $\mathrm{Circ}(d, d/2)$ is empty and has width d*.

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Example

$\mathrm{Vol}(\mathrm{Circ}(4,2)) = 1 + \frac{5}{1}\binom{3}{0}2^2 + \frac{5}{2}\binom{2}{1}2^4 = 1 + 20 + 80 = 101$

$\mathrm{Vol}(\mathrm{Circ}(6,3)) = 1 + \frac{7}{1}\binom{5}{0}3^2 + \frac{7}{2}\binom{4}{1}3^4 + \frac{7}{3}\binom{3}{2}3^6 = \cdots = 6301$

These volumes coincide with those of the unique widest empty 4-simplex ($\mathrm{Cycl}(101,4)$) and the smallest among the six cyclotomic empty 6-simplices of width 6 ($\mathrm{Cycl}(6301,6)$).

This led us to conjecture that *for every even d, the simplex $\mathrm{Circ}(d, d/2)$ is empty and has width d*. It turns out we have been able to prove much more than this.

Width

Theorem

For even d and every m, the width of $\mathrm{Circ}(d, m)$ equals $2m$.

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Proof.

Consider a functional $w \in \mathbb{Z}^{d+1}$, and let $v^{(i)}$ denote the i-th vertex. We have

$$\langle w, v^{(i)} \rangle = m(w_{i+1} - w_{i-1}) + w_i.$$

Since $d + 1$ is odd, we go through all the is in steps of two. Thus, there exist indices i_0 and j_0 with $w_{2i_0} = \min(w)$ and

$$w_{2(i_0-1)} > w_{2i_0} = w_{2(i_0+1)} = \cdots = w_{2j_0} < w_{2(j_0+1)}.$$

(It could well be that $i_0 = j_0$, but this is ok). We distinguish two cases: □

Proof. (cont).

- Case 1: $w_{2i_0-1} \leq w_{2j_0+1}$. Then:

$$\begin{aligned}\langle w, v^{(2j_0+1)}\rangle - \langle w, v^{(2i_0-1)}\rangle = \quad & m\,\underbrace{\left(w_{2j_0+2} - w_{2j_0}\right)}_{\geq 1} \\ & - m\,\underbrace{\left(w_{2i_0} - w_{2i_0-2}\right)}_{\leq -1} \\ & + \underbrace{w_{2j_0+1} - w_{2i_0-1}}_{\geq 0} \qquad \geq 2m.\end{aligned}$$

- Case 2: $w_{2i_0-1} > w_{2j_0+1}$. (Slightly more complicated). $\square$

Emptyness

Theorem

The following conditions are equivalent for the circulant simplex $\mathrm{Circ}(d, m)$*, with* d *even:*

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1. $\mathrm{Circ}(d,m)$ *is an empty simplex.*
2. $\mathrm{Circ}(d,m)$ *does not contain the standard basis vectors.*
3. *m is smaller than the unique positive solution of*

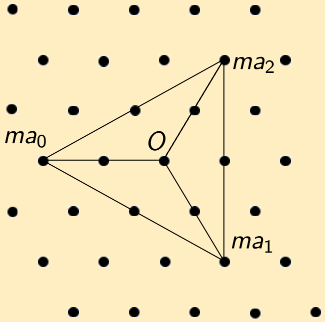
$$m^{d-1} = \sum_{i=0}^{d/2-1} \binom{d-1-i}{i} m^{2i}.$$

4. $m < \frac{1}{2\sinh\alpha}$, *where α is the unique solution of* $\cosh\alpha = \sinh(d\alpha)$.

Emptyness

Idea of proof of 2 $\Rightarrow$ 1.

Consider the vectors $a_i = e_{i+1} - e_{i-1}$, $i = 0, \ldots, d$, so that the vertices of $\mathrm{Circ}(d, m)$ are $v_i := e_i + ma_i$. Let $A = \mathrm{conv}(a_0, \ldots, a_d)$.



The simplex mA, decomposed into $d + 1$ dilated unimodular simplices.

Emptyness

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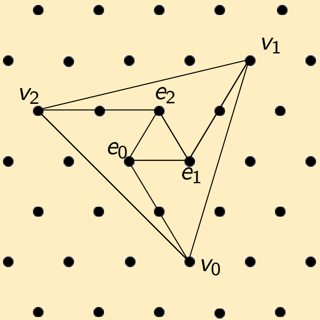

Our simplex Circ(d, m)

Emptyness

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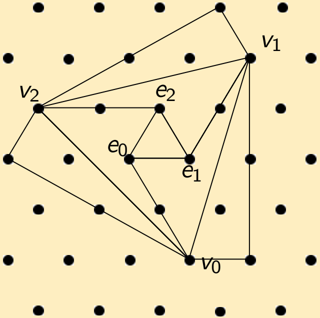

Our simplex $\mathrm{Circ}(d, m)$ is covered by the Minkowski sum $\Delta_0 + mA$.

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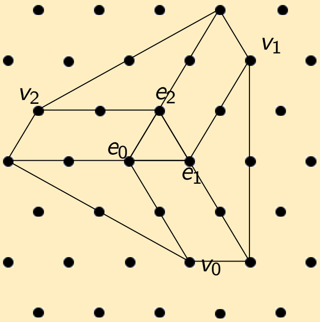

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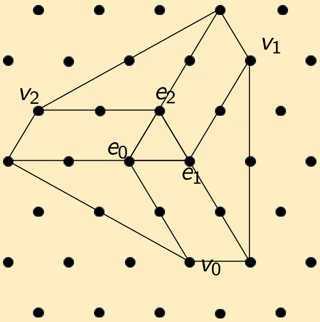
Idea of proof of $2 \Rightarrow 1$.



Our simplex $\mathrm{Circ}(d, m)$ is covered by the Minkowski sum $\Delta_0 + mA$, which decomposes nicely as a mixed subdivision. All lattice points of $\Delta_0 + mA$ lie in the dilated unimodular simplices coming from mA. It can be directly checked (working out the facet description of $\mathrm{Circ}(d, m)$) that for m below the threshold all lattice points in these pieces lie outside $\mathrm{Circ}(d, m)$.

Asymptotics

Let $m_0(d)$ be the threshold in the previous theorem:

Theorem

$$\lim_{d\to\infty} \frac{2m_0(d)}{d} = \frac{1}{\operatorname{arc\,sinh}(1)} \simeq 1.1346.$$

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Corollary

In arbitrary (even) dimension d there are empty lattice simplices of width $\approx 1.1346d$.

Final remarks

- The width we achieve *for empty simplices* is very close (only 0.3% off) to the best known lower bound for $\mathrm{Flt}(d)$:
  $\lim_{d\to\infty} \frac{\mathrm{Flt}_d}{d} \geq \frac{2+\sqrt{2}}{3} \sim 1.1381\ldots$

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Question

Does $\mathrm{Flt}(d)$ change (significantly) if restricted to convex bodies/simplices that have a symmetry acting as cyclic permutation of the lattice generators?

Thank you for your attention

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