The universal valuation of Coxeter matroids

A polytopal view

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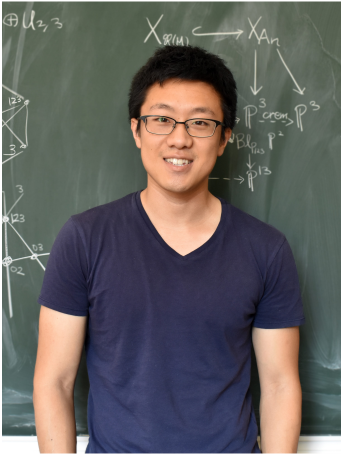
UC Berkeley $\longrightarrow$ KTH

Polytop(ics)
April 8, 2021

Coauthors



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arXiv: 2008.01121

Subdivisions

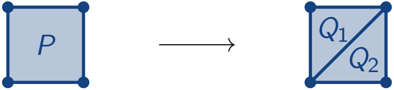
Let $\mathscr{P}$ be a family of polyhedra in V.

Definition

A subdivision of $P \in \mathscr{P}$ is a set $\{Q_1, \ldots, Q_k\} \subseteq \mathscr{P}$ such that

1. $\forall i \;\dim P = \dim Q_i$,
2. $\forall i$ the vertices of Q_i are vertices of P,
3. $Q_1 \cup \cdots \cup Q_k = P$, and
4. $\forall i \neq j$ if $Q_i \cap Q_j$ is nonempty, then it is a proper face of both Q_i and Q_j.

Example: $\mathscr{P} = \{\text{all polyhedra in } \mathbb{R}^2\}$



What is a valuation?

For $I \subseteq [k]$, let $Q_I = \bigcap_{i \in I} Q_i$.

Definition *Polytope version!*

A function $f : \mathscr{P} \to A$ (abelian group) is valuative if for any subdivision $\{Q_1, \ldots, Q_k\}$ of $P \in \mathscr{P}$ the following relation holds:

$$f(P) = \sum_{\varnothing \subsetneq I \subseteq [k]} (-1)^{\dim P - \dim Q_I} f(Q_I)$$

Examples:

- Euclidean volume of polytopes
- Ehrhart polynomials of lattice polytopes
- Tutte polynomials of matroids
- Order polynomials of posets
- Chow classes of generalized permutahedra

What is a valuation?

Alternatively, for $P \in \mathscr{P}$ define $\mathbb{1}_P : V \to \mathbb{Z}$ by

$$\mathbb{1}_P(x) = \begin{cases} 1, & x \in P \\ 0, & \text{otherwise} \end{cases}$$

and let $\mathbb{I}(\mathscr{P})$ be the $\mathbb{Z}$-module of indicator functions

$$\mathbb{I}(\mathscr{P}) := \left\{ \sum_{P \in \mathscr{P}} a_P \mathbb{1}_P \;\middle|\; a_P \in \mathbb{Z},\ \text{finitely many } a_P\text{'s nonzero} \right\}.$$

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Definition *Commutative diagram version!*

A function $f : \mathscr{P} \to A$ is a valuation if there exists a $\mathbb{Z}$-linear map $\tilde{f} : \mathbb{I}(\mathscr{P}) \to A$ such that $f(P) = \tilde{f}(\mathbb{1}_P)$.

Think: $\mathbb{I}(\mathscr{P})$ "models" valuative-ness since

$$\mathbb{1}_P + \mathbb{1}_Q = \mathbb{1}_{P \cup Q} + \mathbb{1}_{P \cap Q}.$$

Universal valuation

$$\begin{tikzcd} \mathscr{P} \arrow[r, "P \mapsto \mathbb{1}_P"] \arrow[dr, "\text{Valuation } f"'] & \mathbb{I}(\mathscr{P}) \arrow[d, "\mathbb{Z}\text{-linear } \tilde{f}"] \\ & \text{Abelian group } A \end{tikzcd}$$

Universal valuation

Universal valuation $\mathcal{F}$: For any valuation f there exists a unique φ such that $f = \varphi \circ \mathcal{F}$.

$$\begin{tikzcd}
\mathscr{P} \arrow[r, "P \mapsto \mathbb{1}_P"] \arrow[rr, bend left, "\text{Universal valuation } \mathcal{F}"] \arrow[dr, "\text{Valuation } f"'] & \mathbb{I}(\mathscr{P}) \arrow[r, "\text{Isomorphism } \tilde{\mathcal{F}}"] \arrow[d, "\mathbb{Z}\text{-linear } \tilde{f}"] & \mathbb{Z}\{\text{basis of } \mathbb{I}(\mathscr{P})\} \arrow[dl, "\exists! \varphi"] \\
& \text{Abelian group } A &
\end{tikzcd}$$

How to construct $\mathcal{F}$: Choose a basis of $\mathbb{I}(\mathscr{P})$

$$P \overset{\mathcal{F}}{\longmapsto} \text{Expression for } \mathbb{1}_P \text{ in this basis}$$

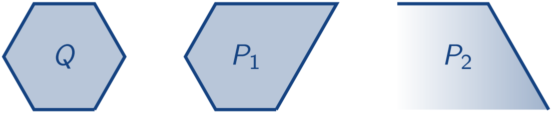
Deformations

We will focus on the case where $\mathscr{P} = \mathrm{Def}(Q)$, the collection of deformations of some polytope $Q \subset V$.

Definition

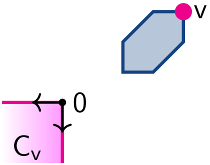
A polyhedron $P \subseteq V$ is a deformation of Q if the normal fan of P coarsens a subfan of the normal fan of Q.

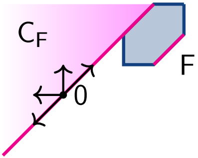
Example: A polytope Q and deformations $P_1, P_2 \in \mathrm{Def}(Q)$



Two definitions by picture

Tangent cone of a polytope at a face:





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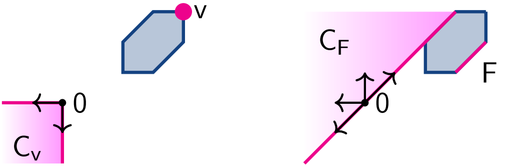


Tight containment of a polyhedron in a cone:

Examples

Non-examples

Universal valuation of deformations

Let $Q \subseteq V$ be a polytope.

Proposition [Eur–Sanchez–S. 2020]

Translated tangent cones of Q form a basis for $\mathbb{I}(\mathrm{Def}(Q))$:

$$T := \{\mathbb{1}_{C+v} | C \text{ is a tangent cone of } Q,\ v \in V\}.$$

Theorem [Eur–Sanchez–S. 2020]

The universal valuation of $\mathrm{Def}(Q)$ is $\mathcal{F} : \mathrm{Def}(Q) \to \mathbb{Z}^T$ given by

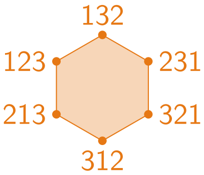
$$\mathcal{F}(P) = \sum_{\substack{C+v \text{ tightly} \\ \text{contains } P}} \mathrm{e}_{C+v}$$

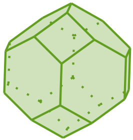
where the $C + v$ are translated tangent cones of Q.

Generalized permutahedra

Definition

The *n-permutahedron* Π_n is the convex hull of all permutations of the coordinates of $(1, 2, \ldots, n) \in \mathbb{R}^n$.





Definition

A *generalized permutahedron* is an element of $\mathrm{Def}(\Pi_n)$.
Equivalently, it is a polyhedron with edge and ray directions of the form $e_i - e_j$.

Coxeter combinatorics

Coxeter combinatorics: Consider combinatorial objects associated to finite reflection groups other than S_n

Definition

Let W be a finite group obtained from reflecting across hyperplanes in V. Let $R \subset V$ be the collection of normal vectors of those hyperplanes. We call the pair $\Phi = (V, R)$ a root system.

$$
\begin{aligned}
A_{n-1} &= (\mathbb{R}^n/(1,\ldots,1),\ \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}) \\
B_n &= (\mathbb{R}^n,\ \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}) \\
C_n &= (\mathbb{R}^n,\ \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : 1 \leq i \leq n\}) \\
D_n &= (\mathbb{R}^n,\ \{\pm e_i \pm e_j : 1 \leq i < j \leq n\})
\end{aligned}
$$

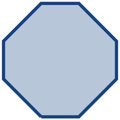
Generalized Coxeter permutahedra

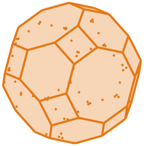
Let $\Phi = (V, R)$ be a root system with reflection group W.

Definition

The Φ-permutahedron Π_Φ is the convex hull of the W-orbit of a "generic" point in V.

Example: B_2- and B_3-permutahedra





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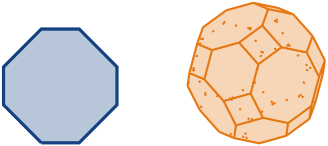

Definition

A generalized Φ-permutahedron is an element of $\mathrm{Def}(\Pi_\Phi)$. Equivalently, it is a polyhedron with edge and ray directions in the set of roots R.

Universal valuation of generalized Φ-permutahedra

Corollary [Derksen–Fink 2010 (Type A), Eur–Sanchez–S. 2020]

The universal valuation of generalized Φ-permutahedra is given by

$$\mathcal{F}(P) = \sum_{\substack{C+v \text{ tightly} \\ \text{contains } P}} \mathrm{e}_{C+v}$$

where the $C + v$ are translated tangent cones of the standard Φ-permutahedron.

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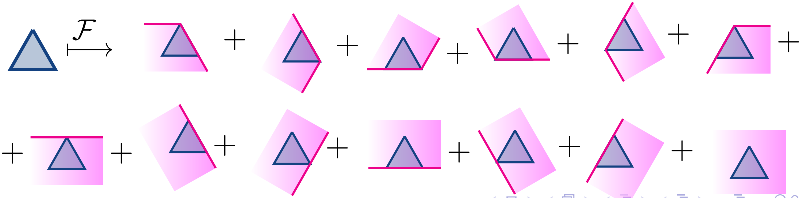
Example: Translated tangent cones of the hexagon (Π_3)



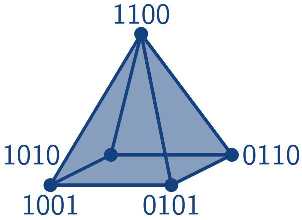
Matroids

Matroids are combinatorial objects that generalize the notion of independence. They are a subfamily of $\mathrm{Def}(\Pi_n)$.

Definition *Polytope version!* [GGMS 1987]

A matroid is a polytope with edge directions of the form $e_i - e_j$ and vertices in $\{0,1\}^n$.

Example:



Uniform matroids

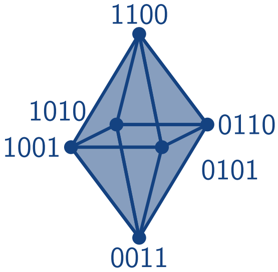
All vertices of a matroid have the same number of 1's, which gives the rank.

Definition

The uniform matroid $U_{r,n}$ is the convex hull of $\{v \in \{0,1\}^n : |v| = r\}$.

All matroids of rank r are contained in $U_{r,n}$!

Example: $U_{2,4}$



Matroid valuations

Recall: The universal valuation of generalized permutahedra is

$$\mathcal{F}(P) = \sum_{\substack{C+v \text{ tightly} \\ \text{contains } P}} \mathrm{e}_{C+v}$$

for translated tangent cones $C + v$ of the permutahedron.

- Let's evaluate $\mathcal{F}$ on a matroid M of rank r
- Since $M \subseteq U_{r,n}$, we don't need to think about the entire cone $C + v$
- What is $(C + v) \cap U_{r,n}$? A Schubert matroid (up to permutation)

Definition

Let $v \in \{0, 1\}^n$ with r 1's. The Schubert matroid Ω_v is the convex hull of all $u \in \{0, 1\}^n$ with r 1's that are lexicographically $\geq v$.

Valuative matroid invariants

Since matroids are generalized permutahedra, they have a natural S_n-action.

Definition

A valuative invariant is a matroid valuation f such that $f(\sigma \cdot M) = f(M)$ for all matroids M and all $\sigma \in S_n$.

Theorem [Derksen–Fink 2010]

The universal valuative matroid invariant is given by

$$\mathcal{G}(M) = \sum_{\substack{\sigma \cdot \Omega_v \supseteq M \\ \text{for some } \sigma \in S_n}} \mathrm{e}_{\Omega_v}.$$

Coxeter matroids

Let $\Phi = (V, R)$ be a root system with reflection group W. Root systems come with special points called fundamental weights.

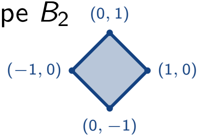
Definition

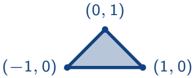
A uniform Φ-matroid is the convex hull of the W-orbit of a fundamental weight.

Definition *Polytope version!* [BGW 2003]

A Φ-matroid is a polytope whose vertices are a subset of the vertices of a uniform Φ-matroid and whose edge directions are roots in R.

Examples: Type B_2

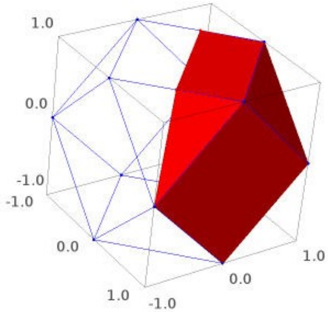




The universal valuative invariant for Φ-matroids

Can we take the same approach as we did in type A? No :(

Example: A uniform matroid of type B_3 intersected with a tangent cone of the B_3-permutahedron



- New vertices
- Bad edge directions
- Not a B_3-matroid
- Not even a generalized B_3-permutahedron!

The universal valuative invariant for Φ-matroids

Nevertheless, our result is analogous to Derksen and Fink's result in Type A! We just needed different proof techniques (0-Hecke algebras).

Definition

Let ϖ be a fundamental weight of Φ and let $w \in W$. The Φ-Schubert matroid Ω_w is the convex hull of $u \cdot \varpi$ such that $u \geq w$ in the Bruhat order.

Theorem [Eur–Sanchez–S. 2020]

The universal valuative Φ-matroid invariant is given by

$$\mathcal{G}(M) = \sum_{\substack{u \cdot \Omega_w \supseteq M \\ \text{for some } u \in W}} \mathsf{e}_{\Omega_w}.$$

Applications

Theorem [Eur–Sanchez–S. 2020]

The interlace polynomial is a specialization of the $\mathcal{G}$-invariant, and hence is a valuative invariant for delta matroids (which are B_n-matroid with vertices in $\{\frac{1}{2}(\pm e_1 \pm \cdots \pm e_n)\}$).

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Theorem [Bastidas, Ardila–Sanchez 2020]

Derksen and Fink's universal valuation for generalized permutahedra is a morphism of Hopf monoids. Using this framework, one can prove that a variety of functions on posets, matroids, and generalized permutahedra are valuative.

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Idea

Our universal valuation for Coxeter GPs should play the same role in Hopf theory for other types as Derksen and Fink's universal valuation does in Type A.

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Thank you!