# Type cones of permutree fans

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(Polytop)ics

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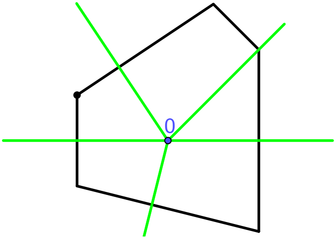
Joint work with V. Pilaud & J. Ritter

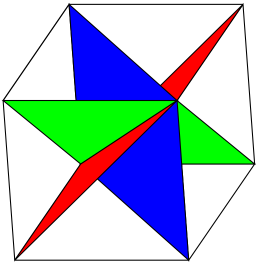
## Definition

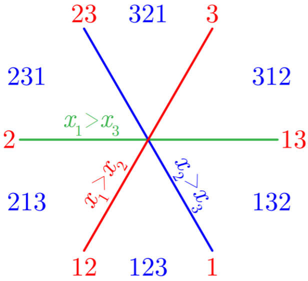
A *fan* is a collection of polyhedral cones, whose faces are in the collection, such that the intersection of any two cones is still a cone in the collection.
The *normal outer cone* of a face of a polytope is the cone spanned by the normal vectors of the facets containing it.
The *normal outer fan* of a polytope is the collection of its faces outer cones. It is a fan. We say that the polytope *realizes* the fan.



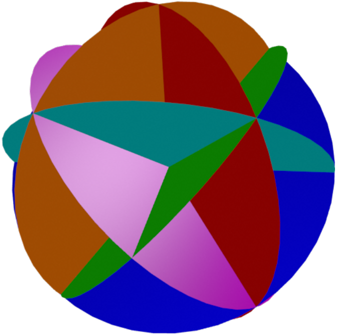
## Definition

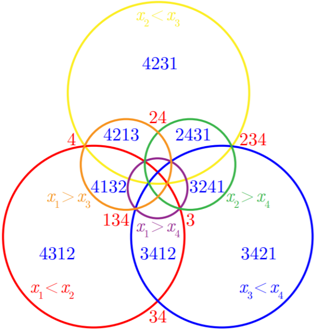
The braid fan $\mathcal{B}_n$ of size $n \in \mathbb{N}$ is the fan induced by the hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$ of $\mathbb{R}^n$ intersected with $\mathbb{H}_n = \left\{ \sum_{i \in [n]} x_i = \binom{n+1}{2} \right\}$.





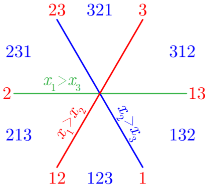
Usefull trick : for $n = 4$, we intersect the fan with a unit sphere, that we stereographically project on the plane.

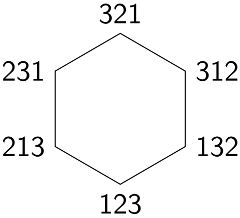




## Definition

The permutahedron is defined as the convex hull of the points $\{(\sigma_i)_{i\in[n]} \mid \sigma \in \mathfrak{S}_n\}$. It is contained in $\mathbb{H}_n$, and the braid fan is its outer normal fan.



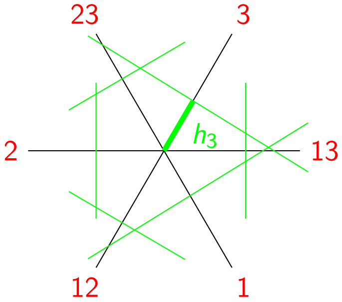

| Braid fan | Permutahedron | Combinatorics |
|---|---|---|
| chamber | vertex | permutation |
| ray | facet | ordered bipartition |
| $k$-dimensional cone | $n-k-1$-dimensional face | ordered partition in $k+1$ parts |

## Question

Given a polytopal fan, what are all the polytopes that realize it ?

## Definition (McMullen 1973)

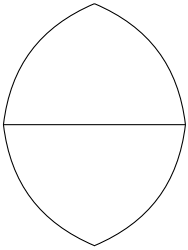
The *type cone* $\mathbb{TC}(\mathcal{F})$ of a fan $\mathcal{F}$ with rays set $\boldsymbol{R}$ is the cone of points $h \in \mathbb{R}^{\boldsymbol{R}}$ such that the polytope defined by the half-spaces $\langle x, r \rangle \leq h_r$ for $r \in \boldsymbol{R}$ has $\mathcal{F}$ as outer normal fan.



## Proposition (Chapoton, Fomin, Zelevinsky 2002)

*If $\mathcal{F}$ is a simplicial (essential, complete) fan, a vector $h$ is in $\mathbb{TC}(\mathcal{F})$ if and only if it satisfies the wall-crossing inequalities for all pairs of adjacent chambers.*

Consider two adjacent chambers of $\mathcal{F}$ with sets of rays $R$ and $R'$ such that $R \setminus \{r\} = R' \setminus \{r'\}$.




There is a unique relationship $\sum_{s \in R \cup R'} \alpha_{R,R'}(s) \cdot s = 0$, with $\alpha_{R,R'}(r) + \alpha_{R,R'}(r') = 2$.

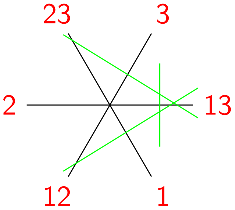
The associated wall-crossing inequality is $\sum_{s \in R \cup R'} \alpha_{R,R'}(s) \cdot h_s > 0$

## Proposition

*In the case of the braid fan, the relations between rays are given by:* $\forall I, J \subseteq [n], r(I) + r(J) = r(I \cup J) + r(I \cap J)$.
*Thus,* $\mathbb{TC}(\mathcal{B}_n)$ *is the set of functions* $h : 2^{[n]} \to \mathbb{R}_{\geq 0}$ *such that:*

$$\begin{cases} h(\emptyset) = h([n]) = 0, \\ h(I) + h(J) > h(I \cup J) + h(I \cap J) \end{cases}$$

*These are called submodular inequalities.*
*Facets of* $\mathbb{TC}(\mathcal{F})$ *are given by submodular inequalities where* $|I \setminus J| = |J \setminus I| = 1$.




$$h_1 + h_3 > h_{13}.$$

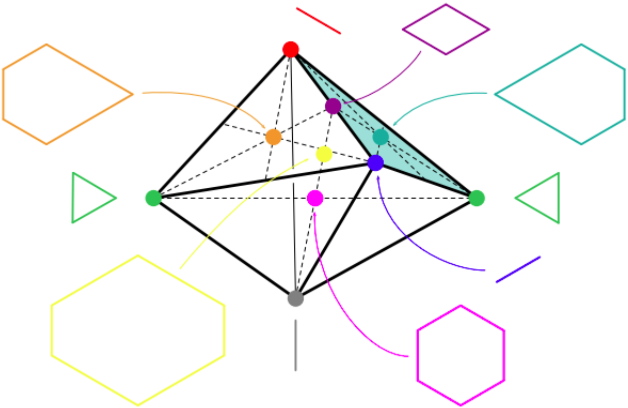
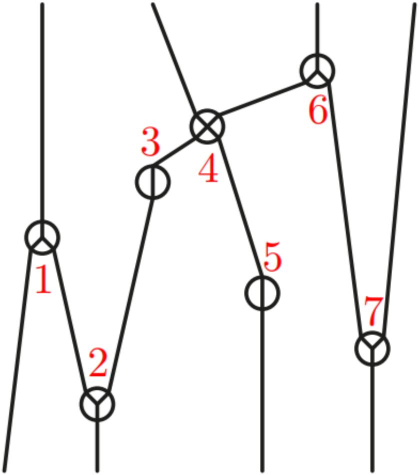


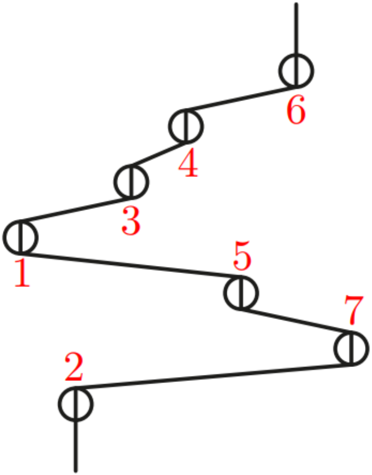
Figure: Section of the type cone of the braid fan of size 3 corresponding to the collection of all deformed permutahedra up to rescaling. (Padrol, Pilaud, Ritter)

## Permutrees

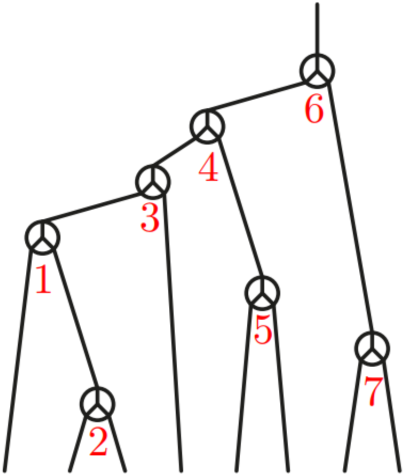



$\delta =$ 

## Permutrees

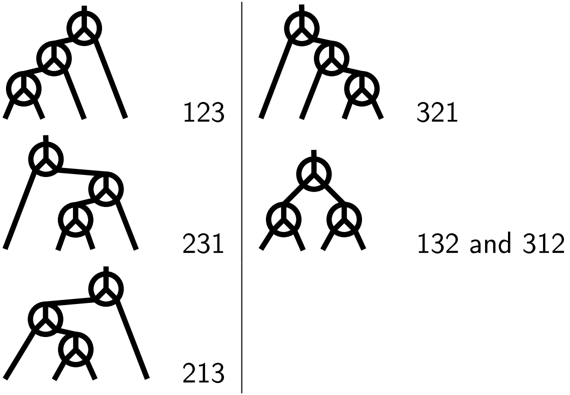



$\delta =$ ⦶⦶⦶⦶⦶⦶⦶



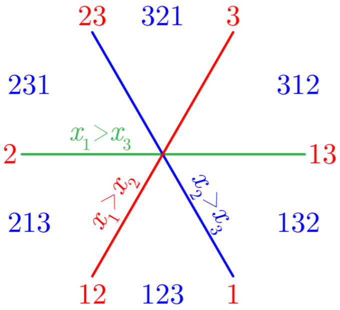

$\delta =$ ☮☮☮☮☮☮☮

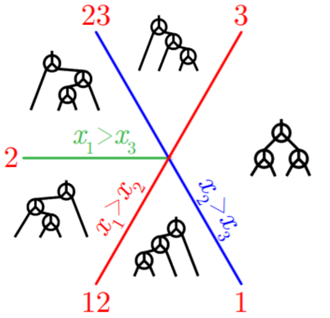
Fix a decoration $\delta \in \{⦶, ☮, Ⓨ, ⊗\}^n$, for instance ☮☮☮.
To every $\delta$-permutree, associate its linear extensions.




This yields an equivalence relation on permutations of size $n$, *i.e.* chambers of the braid fan of size $n$.

Now glue together all chambers of the braid fan of a same class :

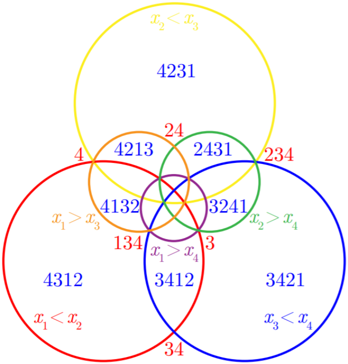


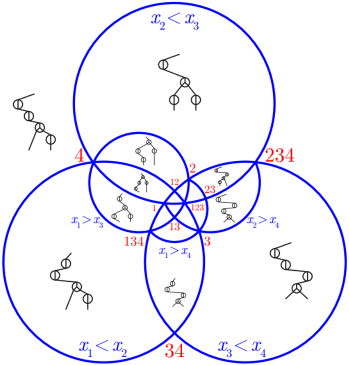




**Definition (Pilaud, Pons 2018)**

This is the $\delta$-permutree fan.

And a bigger example :



**Proposition (Pilaud, Pons 2018)**

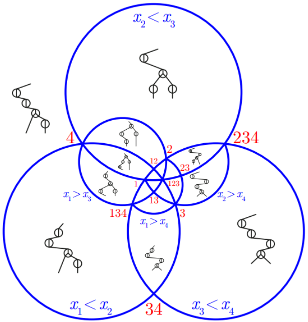
*All $\delta$-permutree fans are polytopal.*

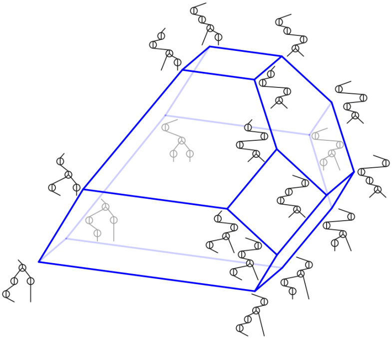
*Sketch of proof:*
Let $\mathcal{I}_\delta$ be the set of rays of the $\delta$-permutree fan.
Call $\delta$-permutreehedron the polytope obtained from inequalities of the permutahedron corresponding to rays in $\mathcal{I}_\delta$ :

$$P_\delta = \bigcap_{I \in \mathcal{I}_\delta} \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i \in I} x_i \geq \binom{|I|+2}{2} \right\} \cap \mathbb{H}_n.$$

Check the wall-crossing inequalities. □

## Polytopes



### Definition

We built the $\delta$-permutreehedron by selecting a subset of the hyperplanes of the facet description of the permutahedron.
Such polytopes are *removahedra*.
Their fans are *removahedral*.

### Proposition (A., Pilaud, Ritter 2021+)

*The permutree fans are removahedral for any realization of the braid fan.*

### Proposition (A., Pilaud, Ritter 2021+)

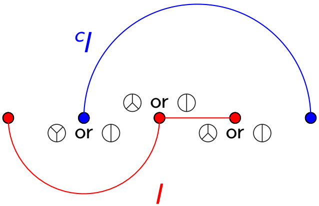
*The only quotient fans of the braid fan that are removahedral are the permutree fans.*

## Proposition (A., Pilaud, Ritter 2021+)

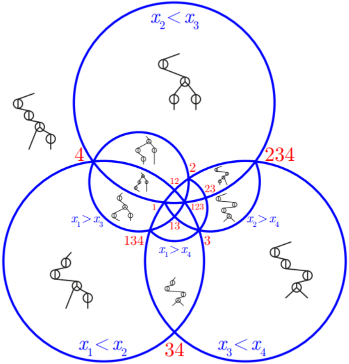
*Given a decoration $\delta$, we call $\delta^+$ (resp. $\delta^-$) the indices of Ⓨ and $\otimes$ (resp. ☮ and $\otimes$) in $\delta$. The following are equivalent:*

- *The ray $r(I)$ for $I \subseteq [n]$ is a ray of the $\delta$-permutree fan.*
- *The bipartition $I, {}^cI$ is an edge cut of a $\delta$-permutree.*
- *For all $a < b < c$, if $a, c \in I$ then $b \notin \delta^- \setminus I$, and if $a, c \notin I$, then $b \notin \delta^+ \cap I$.*

Example: $I = 134, n = 5$.



Back to our example...



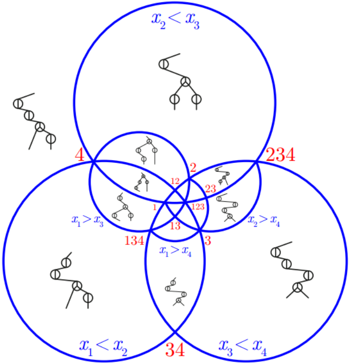

$\delta =$ ⦶⦶☮⦶,
$\mathcal{I}_\delta = \{4, 234, 2, 13, 23, 123, 1, 13, 3, 134, 34\}.$
$\rho(\delta) := |\mathcal{I}_\delta| = 11$

**Proposition (A., Pilaud, Ritter 2021+)**

*Given a decoration $\delta$, we have a simple way to find the pairs of rays defining a facet of the type cone of the $\delta$-permutree fan.*

$$\left( \begin{array}{l} \text{The rays } I \text{ and } J \text{ define a facet of the type cone of the } \delta\text{-permutree fan if and only if, up to swapping } I \text{ and } J\text{:} \\ \begin{cases} i := max(I \setminus J) < min(J \setminus I) =: j \\ I \setminus J = \{i\} \text{ or } \delta_i = \otimes \\ J \setminus I = \{j\} \text{ or } \delta_j = \otimes \\ ]i,j[ \cap \delta^- \subseteq I \cap J \text{ and } ]i,j[ \cap \delta^+ \cap I \cap J = \emptyset \end{cases} \end{array} \right)$$

Back to our example...




$\delta =$ ΦΦ☮Φ,
Pairs of rays defining a facet of $\mathbb{TC}(\delta)$:
$\{\{1,2\}, \{1,3\}, \{12,13\},$
$\{12,23\}, \{12,234\}, \{123,134\},$
$\{123,234\}, \{13,23\}, \{13,34\},$
$\{134,234\}, \{2,3\}, \{23,34\},$
$\{3,4\}\}.$
$\phi(\delta) := \#\{\text{facets of } \mathbb{TC}(\delta)\} = 12$

## Proposition (A., Pilaud, Ritter 2021+)

*We can deduce counting formulas for the numbers of rays $\rho(\delta)$, and facets of the type cone $\phi(\delta)$.*

$$\rho(\delta) = n - 1 + \sum_{\substack{1 \le i < j \le n \\ \forall i<k<j, \delta_k \neq \otimes}} 2^{|\{i<k<j \,|\, \delta_k = \oplus\}|},$$

$$\phi(\delta) = \sum_{\substack{1 \le i < j \le n \\ \forall i<k<j, \delta_k \neq \otimes}} \Omega(\delta_1 \dots \delta_i)^{\delta_i = \oplus} \cdot 2^{|\{i<k<j \,|\, \delta_k = \oplus\}|} \cdot \Omega(\delta_n \dots \delta_{j+1})^{\delta_j = \oplus}.$$

## Corollary (A., Pilaud, Ritter 2021+)

*The type cone $\mathbb{TC}(\delta)$ is simplicial if and only if $\delta_k \neq \oplus$ for $k \in ]1, n[$.*

**Proposition (Padrol, Palu, Pilaud, Plamondon 2019)**

*Let $\mathcal{F}$ be a simplicial fan with $N$ rays whose type cone is simplicial. Let $K$ be the $(N-n) \times N$ matrix whose rows are the inner normal vectors of the facets of $\mathbb{TC}(\mathcal{F})$.*
*Then the polytope $\boldsymbol{Q}(u) := \{s \in \mathbb{R}^N_{\geq 0} \mid Kz = u\}$ is a realization of $\mathcal{F}$ for any $u \in \mathbb{R}^{N-n}_{>0}$.*
*Moreover, all realizations of $\mathcal{F}$ are of this form.*

**Corollary (A., Pilaud, Ritter 2021+)**

*As soon as $\forall k \in ]1, n[, \delta_k \neq$ ⦶, we can give an explicit description of $\mathbb{TC}(\delta)$.*

Let $\delta \in \{\Ydown, \Yup, \otimes\}^n$, with $\delta_1 = \delta_n = \otimes$.
$\mathfrak{F} = \{1 \le i < j \le n \,|\, \forall i < k < j, \delta_k \neq \otimes\}$.
$\mathfrak{R} = \{0,1\} \times [n]^2 \times \{0,1\}$.
$\forall (i,j) \in \mathfrak{F}, \epsilon \in \{+,-\}$:

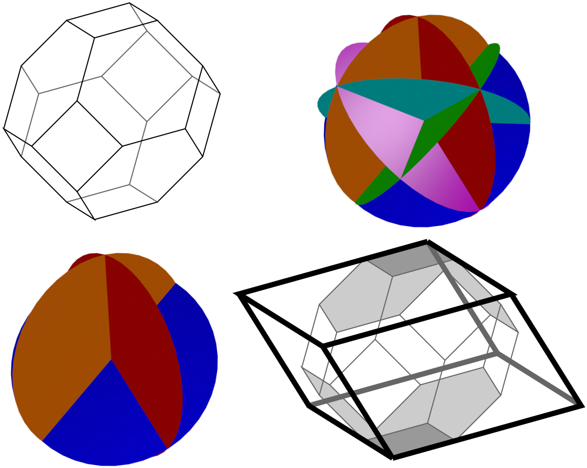
$$p^{\epsilon}_{i,j} = \begin{cases} min(\{j\} \cup (]i,j[ \cap \delta^{\epsilon})) - 1 & \text{if } i \in \delta^{\epsilon}, \\ i-1 & \text{if } i \notin \delta^{\epsilon}. \end{cases}$$

$$q^{\epsilon}_{i,j} = \begin{cases} max(\{i\} \cup (]i,j[ \cap \delta^{\epsilon})) - 1 & \text{if } j \in \delta^{\epsilon}, \\ j+1 & \text{if } i \notin \delta^{\epsilon}. \end{cases}$$

$\forall u \in \mathbb{R}^{\mathfrak{F}}_{>0}$, $\boldsymbol{Q}_\delta(u)$ is defined by:

$$\left\{ z \in \mathbb{R}^{\mathfrak{R}}_{\ge 0} \;\middle|\; \begin{array}{l} z_{(l,p,q,r)} = 0 \text{ if } (p,q) \notin \mathfrak{F}, \quad z_{(l,p,q,r)} = z_{(l',p,q,r')} \text{ if } p+1 \neq q, \\ \text{and } \forall (i,j) \in \mathfrak{F}, z_{(1,p^+_{i,j},q^-_{i,j},0)} + z_{(0,p^-_{i,j},q^+_{i,j},1)} \\ \qquad -z_{(i \notin \delta^-, p^-_{i,j+\mathbf{1}}, q^-_{i-\mathbf{1},j}, j \notin \delta^-)} \\ \qquad -z_{(i \in \delta^+, p^+_{i,j+\mathbf{1}}, q^+_{i-\mathbf{1},j}, j \in \delta^+)} = u_{(i,j)} \end{array} \right\}$$

# Perspectives: type cones of other quotient fans



# Thank you!