Algebraic degrees of 3-dimensional polytopes

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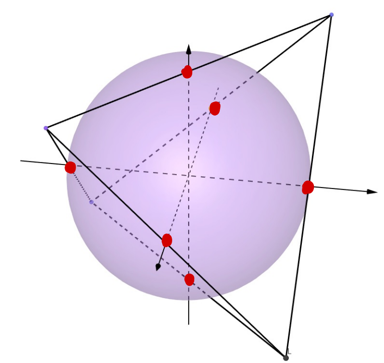
Introduction

Two polytopes are combinatorially equivalent if their faces lattices are isomorphic.

The realization space of a polytope P is the set of all polytopes combinatorially equivalent to P.

In this talk:

- P is a 3-dimensional polytope.
- constrained realizations.



A bit of history

(Some) facts on 3-polytopes:

- Realization spaces are contractible. | Steinitz 1922
- Realizations with rational vertices.
- Realizations with edges tangent to the sphere, | Koebe '36, Andreev '70, Thurston '82, Schramm '91
- ... and such that the origin 0 is the barycenter of the tangency points. | Springborn '05

$(1, 3) \neq (m, d > 2)$

Koebe realizations

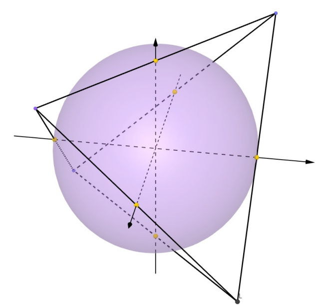
3-polytope P

Koebe realization of P: Q is combinatorially equivalent to P, edges of Q are tangent to unit sphere S^2.

Contact points = tangency points

→ K-A-T: unique repres. modulo admissible proj. transf which fix S^2

Example: tetrahedron



Springborn realization

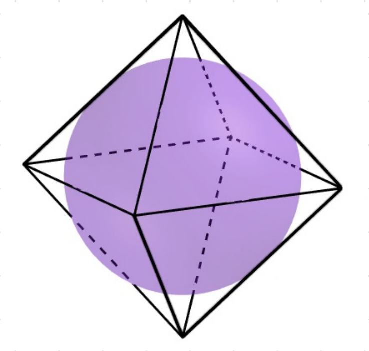
$$\text{vertices} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Springborn realizations

Springborn realization of P: Q is a Koebe realization and the barycenter of the contact pts is the origin.

→ S: unique represent. modulo rotations and reflections

Example: octahedron



$$\text{vertices} = \begin{pmatrix} \pm\sqrt{2} & 0 & 0 \\ 0 & \pm\sqrt{2} & 0 \\ 0 & 0 & \pm\sqrt{2} \end{pmatrix}$$

Koebe degree [Belotti-Joswig-P '21]

Koebe realization space of P:

Realization space of P + tangency conditions

→ Lemma. Realizations with algebraic coordinates are dense.

$\mathbb{Q}[Q]$ is the field extension of $\mathbb{Q}$ given by vertices of Q.

<u>Koebe degree $\kappa(P)$</u> $:= \min\limits_{\substack{Q \text{ Koebe} \\ \text{realization of } P}} \left|\mathbb{Q}[Q] : \mathbb{Q}\right|$

Examples

Koebe degree $\kappa(P) := \min\limits_{\substack{Q \text{ Koebe} \\ \text{realization of } P}} \left| \mathbb{Q}[Q] : \mathbb{Q} \right|$

Tetrahedron: vertices $= \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $\quad \kappa(P) = 1$

Octahedron: vertices $= \begin{pmatrix} \pm\sqrt{2} & 0 & 0 \\ 0 & \pm\sqrt{2} & 0 \\ 0 & 0 & \pm\sqrt{2} \end{pmatrix}$ $\quad \kappa(P) \leq 2$
$= 2$

Cube $\quad \kappa(P) = 2$

Proposition. Let P^* be the polar of P.
Then $\kappa(P) = \kappa(P^*)$.

Properties

Theorem. Let P be a 3-polytope with a triangular face and $n = \#$ vertices. Then

$$\kappa(P) \leq 2^{2^{3n}}.$$

CAD

Theorem. The Koebe degree of any stacked 3-polytope is one.

Springborn degree [Belotti-Joswig-P '21]

Springborn degree $\sigma(P) := \min_{\substack{Q \text{ Springborn} \\ \text{realization}}} |\mathbb{Q}(Q) : \mathbb{Q}|$

Proposition. Let Q be a Springborn realization of P. The algebraic degree of $\mathrm{vol}(Q)$ divides $\sigma(P)$.

Tetrahedron: vertices $= \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $\sigma(P) = 1$.

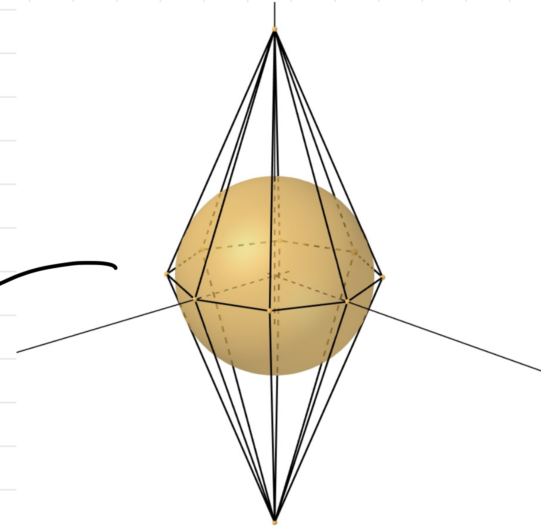
Octahedron: vertices $= \begin{pmatrix} \pm\sqrt{2} & 0 & 0 \\ 0 & \pm\sqrt{2} & 0 \\ 0 & 0 & \pm\sqrt{2} \end{pmatrix}$ $\sigma(P) \leq 2$, $\mathrm{vol}(Q) = \frac{8\sqrt{2}}{3}$ $\Rightarrow \sigma(P) = 2$

Bipyramids

B_k bipyramids over k-gon

$$\frac{\cos(2\pi/k)}{\cos(\pi/k)}$$

$$\frac{\sin(2\pi/k)}{\cos(\pi/k}$$



Theorem. The Springborn and Koebe degrees of a 3-polytope are not bounded by any constant.

Questions

Proposition. Let P be a 3-polytope with m edges.

$$\kappa(P) \leq \sigma(P) \leq \frac{(2m+2)^7}{8} \cdot \kappa(P)$$

Question. Relationship between $\kappa(P)$ and $\sigma(P)$.

Question. Can $\kappa(P)$ be computed from the degree of the cross ratios of contact pts?

Question. Algebraic degrees of higher dimensional polytopes?

Thank you!!