# Congruence Normality of Simplicial Hyperplane Arrangements 

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## Background

A (real, central) hyperplane $H$ is a codimension-1 subspace in $\mathbb{R}^{d}$ :

$$
H:=\left\{x \in \mathbb{R}^{d}: n \cdot x=0 \text { for some } n \in \mathbb{R}^{d}\right\} .
$$

A (finite, real, central) hyperplane arrangement $\mathcal{A}$ is a finite non-empty set of hyperplanes.

Regions $=$ closures of connected components $\mathbb{R}^{d} \backslash \mathcal{A}$.

A region is simplicial if the normal vectors of its facet-defining hyperplanes are linearly independent.
Every region is simplicial $\Rightarrow \mathcal{A}$ is simplicial

The rank of $\mathcal{A}=\operatorname{dim}(\operatorname{span}(\{$ normals $\}))$.

## Grünbaum's list: rank-3 simplicial arrangements 1971/2009


near-pencils

polygons \& symmetries

$2 n$-gons, symmetries, $\infty$


## Grünbaum's list: rank-3 simplicial arrangements 1971/2009



90 Sporadic

$\mathcal{A}(19,1)$

$\mathcal{A}(24,2)$

$\mathcal{A}(37,2)$

## Changes to the Catalogue of Rank 3 Simplicial Arrs.

- 5 new sporadic arrangements (Cuntz 2011, 2020)
- 37 hyperplanes still conjectured upper bound
- The list is complete for up to 27 hyperplanes (Cuntz 2011)
- 53 sporadic arrangements come from finite Weyl groupoids (Cuntz-Heckenberger 2009)

Cuntz-E-Labbé 2020: normals and invariants of the updated catalogue.

Goal: Add more structure to the catalogue
$\rightarrow$ test arrangements for congruence normality in general

## Posets from hyperplane arrangements

Choose a base region $B$, orient all normals away from it:

$$
\mathrm{n} \cdot \mathrm{v}_{B} \leq 0, \quad \mathrm{v}_{B} \in B
$$

The separating set $S(R)$ of a region $R$ is the set of hyperplanes that separate it from $B$.


## Poset of Regions

The poset of regions $P_{B}(\mathcal{A})$ :

- elements: regions
- relations: $R_{1} \leq R_{2}$ if and only if $S\left(R_{1}\right) \subseteq S\left(R_{2}\right)$.



## Congruence normality

What do we know about the poset of regions?

Theorem (Björner-Edelman-Ziegler 1990)
If $\mathcal{A}$ is simplicial, then $P_{B}(\mathcal{A})$ is a lattice for all choices of base region.

Theorem (Caspard-Le Conte de Poly-Barbut-Morvan 2004)
Posets of regions of finite Coxeter arrangements are congruence normal.

## Congruence normality

A subset $S$ of $(P, \leq)$ is convex if $x, y \in S$ and $x \leq z \leq y \Rightarrow z \in S$.

A lattice is congruence normal if it is obtainable from the one element lattice by a finite sequence of doublings of convex sets.


Motivation for studying congruence normality
lattice of regions A simplicial arrangement \& simple zonotope


## Motivation for studying congruence normality



## reworded Conjecture (Padrol-Pilaud-Ritter 2020)

If the poset of regions of a simplicial arrangement is congruence normal, then any lattice quotient is polytopal.

## Classification of the Catalogue of Rank 3 Simplicial Arrs.

| $P_{B}(\mathcal{A})$ always $\mathbf{C N}$ | $P_{B}(\mathcal{A})$ sometimes $\mathbf{C N}$ | $P_{B}(\mathcal{A})$ never $\mathbf{C N}$ |
| :---: | :---: | :---: |
| Finite Weyl Groupoids | $\mathcal{F}_{2}(m)(m \geq 10)$ | $\mathcal{A}(22,288)$ |
| $\mathcal{F}_{2}(m)(m \leq 8)$ | $\mathcal{F}_{3}(m)(m \geq 17)$ | $\mathcal{A}(25,360)$ |
| $\mathcal{F}_{3}(m)(m \leq 13)$ | 41 arrangements | $\mathcal{A}(35,680)$ |
| $\mathcal{A}(15,120)$ |  |  |
| $\mathcal{A}(31,480)$ |  |  |
| 55 arrangements | 61 arrangements | 3 arrangements |

## Checking if an Arrangement is Congruence Normal

Shards are pieces of hyperplanes used to understand lattice congruences.


## Shards: Congruence Normality Criterion



Geometric criterion for creating directed graph on shards.

## Theorem (Reading 2004)

A simplicial arrangement is congruence normal iff the directed graph on shards is acyclic.

## Shards $\rightarrow$ shard covectors

Let $\mathrm{N}=\left\{\mathrm{n}_{i}\right\}_{i \in[m]}=\operatorname{normals}(\mathcal{A})$.
A covector on $N$ is a vector of signs $\left(c_{i}\right)_{i \in[m]} \in\{0,+,-\}^{m}$ :

$$
c:=\left(\operatorname{sign}\left(\mathrm{x} \cdot \mathrm{n}_{i}+\mathrm{a}\right)\right)_{i \in[m]},
$$

where $x \in \mathbb{R}^{d}$ and $a \in \mathbb{R}$.

A restricted covector on $\mathrm{U} \subseteq \mathrm{N}$ has "*"s on all entries not in U .

## Theorem (Cuntz-E-Labbé)

There is a bijection between shards and shard covectors.

## Translation of Shard Cycle Criterion

## Theorem (Reading 2004)

A simplicial arrangement is congruence normal iff the directed graph on shards is acyclic.

## Theorem (Cuntz-E-Labbé)

Let $\Sigma, \Theta$ be shards on hyperplanes $i, j$ respectively, let $\sigma, \theta$ be their shard covectors.

$$
\begin{aligned}
\qquad \rightarrow \Theta \text { if and only if } & \theta_{i} \in\{+,-\} \\
& \text { and } \\
& \exists \text { line covector } h: h \cap \sigma \cap \theta=h .
\end{aligned}
$$

## Thank you for listening

