Generalized scattering amplitudes, matroidal blade arrangements and the positive tropical Grassmannian

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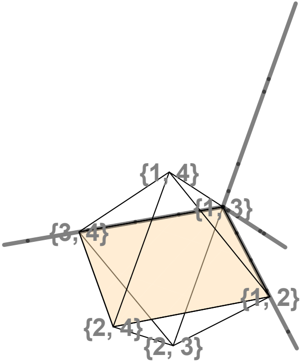
(Polytop)ics: Recent advances on polytopes

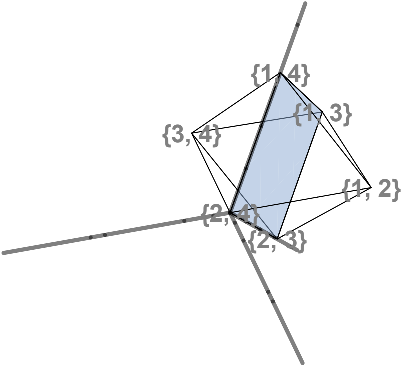
April 6, 2021

Relevant References

- References, from 2018 - 2020, including:
- 2018: [E 1804.05460], [E 1810.03246]
- 2019: [CEGM 1903.08904], [BC 1910.10674], [E 1910.11522], [E 1912.13513].
- 2020: [CE 2003.07958], [E 2005.12305], [CE 2010.09708].

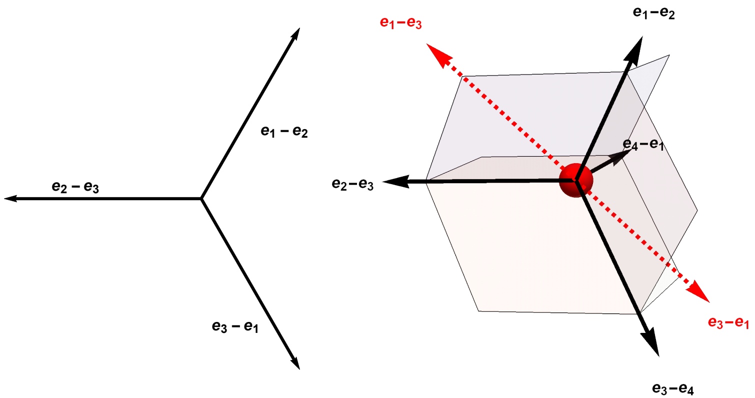
Subdividing Octahedra





The two nontrivial *blade arrangements* $\beta_{1,3}$ (left) and $\beta_{2,4}$ (right) on the vertices $e_1 + e_3$ resp. $e_2 + e_4$ of the octahedron $\Delta_{2,4}$. These are related by the flip $\beta_{1,3} \leftrightarrow \beta_{2,4}$; the blade is simply translated across $\Delta_{2,4}$.

Blades and their rays



Rays (Black arrows) of blades are parallel to roots $e_i - e_{i+1}$. Left: the blade $((1,2,3))$. Right: the blade $((1,2,3,4))$. Red arrows indicate how the blades $((1,2,3))$ and respectively $((1,3,4))$ embed into $((1,2,3,4))$.

Itinerary

- Motivation mixes combinatorial geometry and scattering amplitudes.
- What's known/not known about the biadjoint scalar $m_n^{(2)}(\alpha, \beta)$ and its positive part $m_n^{(2)}$?
- What's in our toolbox to study the *generalized* biadjoint scalar $m_n^{(k)}(\alpha, \beta)$ and its positive part $m_n^{(k)}$?
- Define the (dual) kinematic space, weighted blade arrangements $\mathcal{Z}(k, n)$.
- **Theorem:** the positive tropical Grassmannian $\text{Trop}^+ G(k, n)$ embeds canonically into $\mathcal{Z}(k, n)$; we characterize its image. How does this help to understand the poles of $m_n^{(k)}$?
- Planar cross-ratios, weak separation and binary structures and geometries.

Questions For Today I: Combinatorial Geometries

- Denote $e_J = \sum_{j\in J} e_j$.
- Hypersimplex: for each $1 \le k \le n-1$, a convex polytope $\Delta_{k,n} = \text{conv}\{e_{i_1} + \cdots + e_{i_k} : I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}\}$.
- What makes for a "good" decomposition $\Delta_{k,n} = P_1 \cup \cdots \cup P_m$ into sub-polytopes P_i?
- Some possible criteria:
  1. No new vertices.
  2. No new edges: $\text{edges}(P_i) \subset \text{edges}(\Delta_{k,n})$.
  3. Regular, i.e. induced by projecting down the bends of a continuous, piecewise-linear surface over $\Delta_{k,n}$.
  4. Require internal facets to have the form $\sum_{j\in[a,b]} x_j = r$ where $[a,b] \subset \{1, \ldots, n\}$ is a cyclic interval.
- In our story of weighted blade arrangements, 1 and especially 3, 4 are baked in; only real choice lies in (2).
- Not having to worry about facet normals can be a significant advantage!
- We'll be taking arrangements of a certain cyclically skewed tropical hyperplane, the *blade* $((1,2,\ldots,n))_{e_J}$ on the vertices

Questions For Today II: Scattering Amplitudes

- We explore the poles of certain homogeneous rational functions $m_n^{(k)}$ on the *kinematic space* $\mathcal{K}(k, n) \simeq \mathbb{R}^{\binom{n}{k}-n}$, arising in the study of scattering amplitudes. Here (k, n) are integers satisfying $2 \leq k \leq n-2$.
- History: $m_n^{(2)}$ first studied by Cachazo-He-Yuan [CHY2014], using the so-called biadjoint scalar amplitude $m_n(\alpha, \beta)$ when $\alpha = \beta = (12 \cdots n)$, cyclic orders.
- Generalized to all $k \geq 3$ by Cachazo-E-Guevara-Mizera [CEGM2019].
- We already know a lot about the $m_n^{(k)}$ but many puzzles remain!
- Today we'll see what weighted blade arrangements on the hypersimplex $\Delta_{k,n}$ have to say about the singularities of $m_n^{(k)}$.

What we know about $m_n^{(2)}$

- By now $m_n^{(k=2)}$ is understood quite well. Here's a summary:
  1. $m_n^{(2)}$ has $\binom{n}{2} - n$ simple poles, of the form $t_{i,i+1,\ldots,j}^{-1}$ where $t_{i,i+1,\ldots,j} = \sum_{i\leq a<b\leq j} s_{a,b}$. Here $s_{a,b} = s_{b,a}$ with $s_{a,a} = 0$ and $\sum_{b=1}^{n} s_{a,b} = 0$, are coordinate functions on $\mathcal{K}(2,n)$.
  2. CHY noticed that the poles of $m_n^{(2)}$ form a *basis* of linear functions on $\mathcal{K}(2,n)$.
  3. $m_n^{(2)}$ has exactly Catalan-many $C_{n-2} = 2, 5, 14, \ldots$ maximal collections of compatible simple poles.
  4. Mizera(2018) and Arkani-Hamed et al (2018) identified singularities of $m_n^{(2)}$ with the face poset of the dimension $n-3$ associahedron.
  5. Example:

$$\begin{aligned} m_4^{(2)} &= \frac{1}{t_{12}} + \frac{1}{t_{23}} \\ m_5^{(2)} &= \frac{1}{t_{12}t_{123}} + \frac{1}{t_{12}t_{34}} + \frac{1}{t_{23}t_{123}} + \frac{1}{t_{23}t_{234}} + \frac{1}{t_{34}t_{234}}. \end{aligned}$$

  6. Rem. [E2018], $m_n^{(2)}$ is dual to the *facet deformation cone* of the dim $= n-3$ associahedron (in the Loday representation).

What we know about $m_n^{(k)}$ I: Blades and Generalized Feynman Diagrams

- $m_n^{(k)}$ was introduced by [CEGM]; $m_n^{(3)}$ for $n = 6, 7$ computed two ways:
    1. Via the generalized biadjoint scalar $m^{(k)}(\alpha, \beta)$ and certain generalized scattering equations,
    2. Using a certain polyhedral fan, the *positive tropical Grassmannian* $\mathrm{Trop}^+ G(k, n)$.
- But systematic tabulation of these poles in general is *hard!* Many efforts to tackle this, including...
- In 2019, [Borges-Cachazo], [Cachazo,Guevara,Umbert,Yong]: certain metric tree arrangements define Generalized Feynman Diagrams (GFD's) for $m_n^{(3)}(\alpha, \beta)$ and $m_n^{(k)}(\alpha, \beta)$ resp.
- In 2019/2020 [E]: (matroidal) weighted blade arrangements: dual to GFD's. Construct from blades a certain *planar basis* of kinematic invariants that is essentially cyclically invariant.
- In [Guevara, Yong]: poles, compatibility and certain soft and hard limits for higher k.

What we know about $m_n^{(k)}$ II: related work

- In [Arkani-Hamed,Lam,He]: showed that poles of $m_n^{(k)}$ are dual to *rays* of $\text{Trop}^+ G(k,n)$.
- Cluster algebra approach: [Drummond, Foster, Gurdogan, Kalousios] and [Henke, Papathanasiou],
- Amplituhedra and positroidal subdivisions: [Lukowski, Parisi, Williams],
- Positive tropical Grassmannian [Speyer,Williams].
- Positive configuration spaces, binary geometries, planar $\mathcal{N} = 4$ SYM [Arkani-Hamed et al].

Part 1: Blades Definition.

- Definition [Ocneanu]. Fix an integer $n \geq 3$. The *naturally ordered* blade is the union of the boundaries of n polyhedral cones:

$$\beta = ((1,2,\ldots,n)) = \bigcup_{j=1}^{n} \partial \left\{ \sum_{i \neq j} t_i(e_i - e_{i+1}) : t_i \geq 0 \right\}.$$

- Other interpretations:
- Proposition[E]. This is a tropical variety. It is also the $n-2$ skeleton of the normal fan to the Weyl alcove $x_1 \leq \cdots \leq x_n \leq x_1 + 1$.
- Equivalently, with $h_{ij} = e_i - e_j$, it's the union of the $\binom{n}{2}$ simplicial cones

$$((1,2,\ldots,n)) = \bigcup_{1 \leq i < j \leq n} \text{cone}_+ \left\langle h_{12}, \ldots, \widehat{h_{i,i+1}}, \ldots, \widehat{h_{j,j+1}}, \ldots, h_{n1} \right\rangle.$$

Blades are tropical hypersurfaces

Let $V_0^n \subset \mathbb{R}^n$ be the hyperplane $x_1 + \cdots + x_n = 0$.

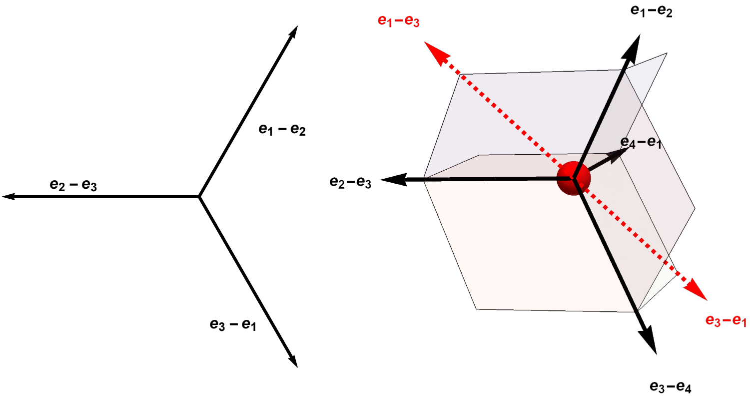
Defn. Let $h : V_0^n \to \mathbb{R}$ be the piece-wise linear function $h(x) = \min\{L_1(x), \ldots, L_n(x)\}$, where

$$L_j = x_{j+1} + 2x_{j+2} + \cdots (n-1)x_{j-1}.$$

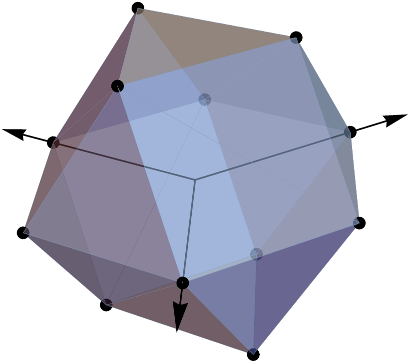
Prop.[E,Oct2019]. The blade $((1,2,\ldots,n))$ equals the bend locus of the function $h(x)$. That is,

$$((1,2,\ldots,n)) = \left\{x \in V_0^n : (L_i(x) = L_j(x)) \leq L_\ell(x) \text{ for all } \ell \neq i,j\right\}.$$

Blades and their rays: a second look

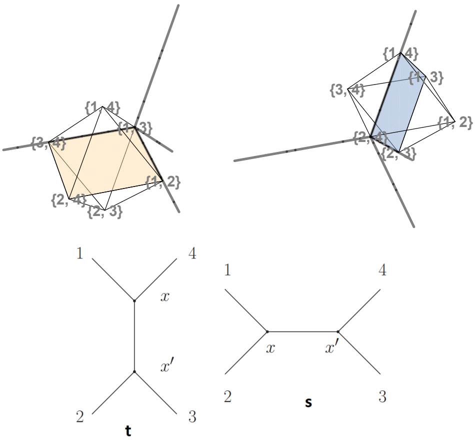


Rays (Black arrows) of blades are parallel to roots $e_i - e_{i+1}$. Left: the blade $((1,2,3))$. Right: the blade $((1,2,3,4))$. Red arrows indicate how the blades $((1,2,3))$ and respectively $((1,3,4))$ embed into $((1,2,3,4))$.



In this talk we use top dimension blades to induce certain subdivisions of *hypersimplices*; but this is not the whole story. Above: the blade $((1,2,3))$ embedded in the hyperplane $x_4 = 0$, depicted inside the root solid: a neighborhood of a point in the type A_3 root lattice. Clearly, the tripod does not induce a subdivision of the ambient space!

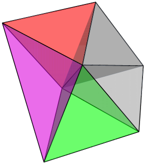
From blades on the Archetypal Octahedron to amplitudes



Dictionary: *Blades* [E2019] ⇔ (tree level) Feynman Diagrams. Left: t-channel Right: *s*-channel. Not shown: *u*-channel.

Associahedron, Root Polytopes, Generalizations

- S. Mizera (2017), and Arkani-Hamed et al (2017) interpreted $m_n(\alpha, \beta)$ in terms of intersecting pairs of associahedra in the moduli space of stable pointed curves $\mathcal{M}_{0,n}$.
- Also, [E2018]:
  1. $m_n^{(2)}((12\cdots n),(12\cdots n))$ blows up exactly on the faces of a particular generalized permutohedron, the dimension $n-3$ associahedron $\mathcal{A}(s)$ in the Loday representation, with given facet deformation parameters s.
  2. $m_n^{(2)}((12\cdots n),(12\cdots n))$ is dual to a certain triangulation of the root polytope $\operatorname{conv}(0,\{e_i - e_j : 1 \leq i < j \leq n-1\})$.



Triangulating $\operatorname{conv}\{0, h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}\}$ where $h_{ij} = e_i - e_j$.

Background physical motivation 1: the Biadjoint Scalar

One has a Lagrangian for the biadjoint scalar amplitude with given flavor group $U(N) \times U(N)$,

$$\mathcal{L}^{\Phi^3} = -\frac{1}{2}\partial_\mu \Phi_{I,\tilde{I}} \partial^\mu \Phi^{I,\tilde{I}} + \frac{\lambda}{3!} f_{I,J,K} \tilde{f}_{\tilde{I},\tilde{J},\tilde{K}} \Phi^{I,\tilde{I}} \Phi^{J,\tilde{J}} \Phi^{K,\tilde{K}},$$

where the $f_{I,J,K}$, $\tilde{f}_{\tilde{I},\tilde{J},\tilde{K}}$ are structure constants for their respective Lie algebras. The $\Phi^{I,\tilde{I}}$ are fields, i.e. certain functions on Minkowski space $\mathbb{R}^{3,1}$.

Standard construction in physics: the biadjoint scalar *amplitude* can be "color decomposed" as

$$M_n = \sum_{\alpha,\beta \in S_n/(\mathbb{Z}/n)} \mathrm{tr}(T^{I_{\alpha_1}} T^{I_{\alpha_2}} \cdots T^{I_{\alpha_n}}) \mathrm{tr}(T^{I_{\beta_1}} T^{I_{\beta_2}} \cdots T^{I_{\beta_n}}) m_n(\alpha, \beta),$$

where the T^{I_j} are certain generators for the "flavor/color" group $U(N)$, and $m_n(\alpha, \beta)$ is the double partial amplitude.

$\Rightarrow$ This talk: we study $m_n(\alpha, \beta)$ (now denoted $m_n^{(2)}$) and its generalization $m_n^{(k)}$ [CEGM2019].

Physical motivation 2: CHY formulation

- Cachazo-He-Yuan [CHY2013] introduced a compact formula to compute scattering amplitudes for a wide variety of Quantum Field Theories. In particular: $m_n^{(2)}(\alpha, \beta)$.
- The CHY construction of $m_n^{(2)}(\alpha, \beta)$ involves a sum over the critical points of a certain log potential function $\mathcal{S} = \sum \log(\Delta_{i,j}) s_{ij}$ on $G(2, n)/T$ and for this talk will remain a black box.
- Let $\{s_{i,j} : i, j = 1, \ldots, n\}$ be variables subject to $s_{i,i} = 0$, $s_{i,j} = s_{j,i}$ and $\sum_{j \neq i} s_{i,j} = 0$.
- Example:

$$m_4((1234), (1234)) = \frac{1}{s_{12}} + \frac{1}{s_{23}},$$

while for $n = 5$ we have

$$\begin{aligned} m_5((12345), (12345)) = & \quad \frac{1}{s_{12}s_{123}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{234}} \\ m_5((12345), (12435)) = & \quad -\frac{1}{s_{3,4}s_{5,1}} - \frac{1}{s_{1,2}s_{3,4}}. \end{aligned}$$

Physical motivation 3: Generalized Feynman Diagrams

- Borges-Cachazo [BC2019] (for $k = 3$) and Cachazo et al [CGUZ2019] (for $k \geq 4$) formulated the Generalized Feynman Diagram expansion for $m_n^{(k)}(\alpha, \beta)$ using collections and then arrays of metric trees. Define

$$m_n^{(k)} = \sum_{\mathcal{C} \in \text{max'l cones } \mathrm{Trop}^+ G(k,n)} \frac{P_{\mathcal{C}}(s)}{Q_{\mathcal{C}}(s)},$$

  where $P_{\mathcal{J}}$ and $Q_{\mathcal{J}}$ are functions on the kinematic space, constructed from rays of $\mathcal{C}$.

- Cachazo-E [CE2020] reformulated $m_n^{(k)}$ as a single integral (which has certain convergence requirements on (s)),

$$mTrop_n^{(k)} = \int_{\mathbb{R}^{(k-1)\times(n-k-1)}} \exp(-\mathcal{F}_{k,n}) dx,$$

  where $\mathcal{F}_{k,n}$ is a certain continuous piece-wise linear function on $\mathbb{R}^{(k-1)\times(n-k-1)}$. Over each linear domain, $mTrop_n^{(k)}$ evaluates to a single contribution $\frac{P_{\mathcal{C}}}{Q_{\mathcal{C}}}$.

Rest of the talk: plan

- From positive tropical Grassmannian $\mathrm{Trop}^+ G(k,n)$ to linear functions on the Kinematic Space.
- Define blades [A. Ocneanu] and their arrangements [E2019].
- **Prop.** Blades induce certain very special matroid subdivisions, called multi-splits. When do two (positroidal) multi-splits have a common matroidal refinement?
- **Thm.** Blade arrangements are matroidal $\Leftrightarrow$ weak separation.
- **Thm.** Blades induce a *basis* for the *dual kinematic space* [E2020], used in collaboration with Cachazo, Guevara, Mizera in scattering amplitudes [CEGM2019; CE2020].
- Planar cross-ratios and binary-type equations on configurations of n points in $\mathbb{CP}^{k-1}$.

Kinematic space $\mathcal{K}(k, n)$

- **Defn.** The kinematic space $\mathcal{K}_{k,n}$ is the following codimension n subspace of $\mathbb{R}^{\binom{n}{k}}$:

$$\mathcal{K}_{k,n} = \left\{ (s) \in \mathbb{R}^{\binom{n}{k}} : \sum_{J : J \ni a} s_J = 0 \text{ for each } a = 1, \ldots, n \right\}.$$

Positive Tropical Plucker vectors

- **Defn.** A vector $\pi = \sum_J c_J e^J \in \mathbb{R}^{\binom{[n]}{k}}$ is said to be a *positive tropical Plucker vector* provided that

$$c_{L\cup\{a,c\}} + c_{L\cup\{b,d\}} = \min(c_{L\cup\{a,b\}} + c_{L\cup\{c,d\}}, c_{L\cup\{a,d\}} + c_{L\cup\{b,c\}})$$

  for any $L \cup \{a,b,c,d\} \in \binom{[n]}{k+2}$ with $a < b < c < d$ cyclically.
- Denote by $\text{Trop}^+ G(k,n)$ the set of all positive tropical vectors.
- **Remark:** The set of positive tropical Plucker vectors is historically called the *positive Dressian*; however, recently ([Speyer,Williams2020] and [Arkani Hamed, Lam, Spradlin2020]) showed that the positive Dressian is equal to the so-called positive tropical Grassmannian.

Dualizing the Positive Tropical Grassmannian $\mathrm{Trop}^+ G(k,n)$

- Consider now the map into the dual kinematic space $(\mathcal{K}(k,n))^*$, $\varphi : \mathrm{Trop}^+ G(k,n) \to (\mathcal{K}(k,n))^*$, the space of linear functions on the kinematic space.

$$\varphi(\pi) = \sum_{\{J\} \in \binom{[n]}{k}} \pi_J s_J.$$

Note: can show that $\ker(\varphi)$ coincides with an n-dimensional subspace of $\mathbb{R}^{\binom{n}{k}}$ that is sometimes called the *lineality space*.

Dual Kinematics Image of $\text{Trop}^+ G(2, n)$

The following result was shown in [E2020] for all k, but let us first formulate $k = 2$: the image in $\mathcal{K}(2, n)^*$ has a simple characterization: positive and noncrossing support in the planar kinematic invariants $\eta_{i,j}$.

Thm[E2020]. For $\eta(\mathbf{c}) = \sum c_J \eta_J$ define $\text{supp}(\eta(\mathbf{c})) = \{\{J\} : c_J \neq 0\}$. Then,

$$\varphi(\text{Trop}^+ G(2, n)) \;=\; \{\eta(\mathbf{c}) : \mathcal{K}(2, n) \to \mathbb{R} : c_{i,j} \geq 0,\ \text{supp}(\eta(\mathbf{c}))\ \text{n.c.}\}$$

Examples. The following are images of ray generators in $\text{Trop}^+ G(2, n)$ for $n = 4, 5$ resp.

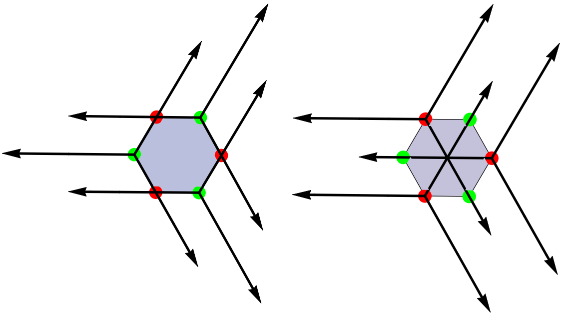
$$\begin{aligned} \eta_{24} &= \frac{1}{4}\left(3s_{12} + 2s_{13} + s_{14} + s_{23} + 3s_{34}\right) \\ &= s_{34} \\ \eta_{25} &= s_{34} + s_{35} + s_{45} \end{aligned}$$

Moral: the coefficients of the s_{ij}'s determine a height function up to lineality!

Part 2

- Now we'll develop the blade arrangement model to help us understand the image of $\varphi(\mathrm{Trop}^+ G(k,n))$, and by extension the singularities of $m_n^{(k)}$.

Blade arrangements: low-dimensional intuition



Two arrangements of the blade $((1, 2, 3))$ on the vertices of a hexagon. Blade arrangement on left induces the trivial subdivision. Blade arrangement on right induces a 6-chamber subdivision.

These are projections of (matroidal) blade arrangements on $\Delta_{3,6}$. The vertex sets are pairwise weakly separated vertices of $\Delta_{3,6}$.

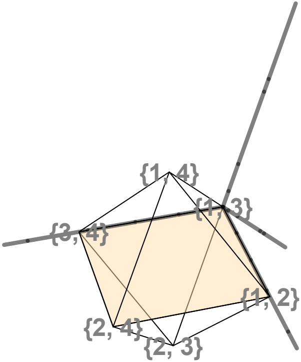
Matroid subdivisions

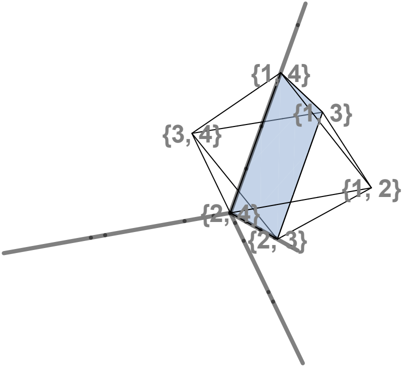
- **Definition.** A *matroid subdivision* of a hypersimplex $\Delta_{k,n}$ is a decomposition $P_1 \cup \cdots \cup P_m = \Delta_{k,n}$ into matroid polytopes P_j, such that each pair $\{P_i, P_j\}$ intersects only on their common face $P_i \cap P_j$. This subdivision is matroidal if each P_i is a matroid polytope: edge directions must be among the *roots* $e_i - e_j$...
- A matroid subdivision is *positroidal* if no octahedral face of $\Delta_{k,n}$ is cut with a hyperplane defined by an equation with crossing indices, e.g. $x_1 + x_3 = x_2 + x_4 = 1$.

Main constructions

- **Definition.** A *blade arrangement* is a superposition of several copies of the blade $\beta = ((1,2,\ldots,n))$, on the integer lattice $\{x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = k\}$ for some fixed integer k.
- However, we shall always consider blade arrangements on the vertices of hypersimplices $\Delta_{k,n} = \{x \in [0,1]^n : \sum_{i=1}^{n} x_i = k\}$ with $1 \leq k \leq n-1$.
- **Definition** [E2019]. A *matroidal* blade arrangement $\beta_{J_1}, \ldots, \beta_{J_m}$ is an arrangement of the blade $\beta = ((1,2,\ldots,n))$ on the vertices $e_{J_1}, \ldots, e_{J_m}$ of $\Delta_{k,n}$ such that every maximal cell is *matroidal*: i.e., all edges of each maximal cell is in a root direction $e_i - e_j$.

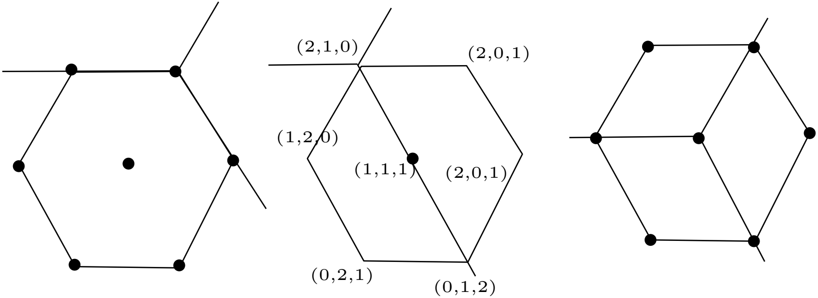
Review 1: Octahedral blade arrangements





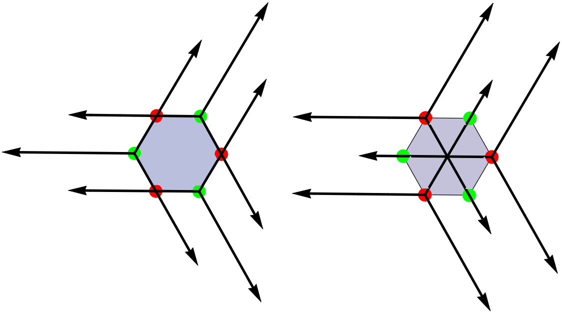
The two nontrivial blade arrangements on the octahedron $\Delta_{2,4}$. Edges of the octahedron are in the directions $e_i - e_j$. Same for the pairs of square pyramids.

Blade Arrangements in three coordinates: 1



1-split, 2-split, 3-split of a hexagon: induced by pinning a single *blade* $((1,2,3))$ to a vertex of a hexagon. Note: here, this once, we allow a new internal vertex!

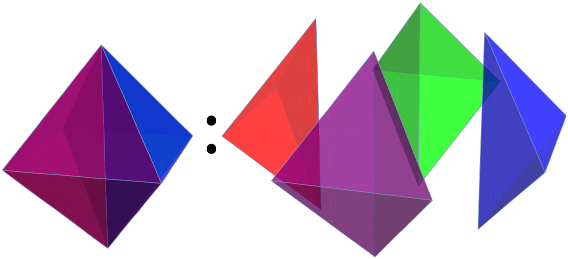
Blade Arrangements: 2



Two arrangements of the blade $((1, 2, 3))$ on the vertices of a hexagon. Blade arrangement on left induces the trivial subdivision. Blade arrangement on right induces a 6-chamber subdivision.

Review 2: Non-matroidal subdivision of the octahedron

- Three ways to split the octahedron into two half-pyramids, along the three equatorial planes.
- Any two at a time induces a triangulation of the octahedron into four tetrahedra.



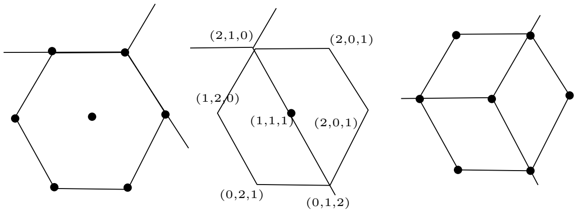
Example: the four tetrahedra now share an edge, the diagonal direction, say $e_1 + e_4 - e_2 - e_3$ (equivalently, their vertex sets don't define matroids).

2-splits of $\Delta_{2,n}$, d-splits of $\Delta_{k,n}$

- **Defn/Example.** A *2-split* (of $\Delta_{2,n}$) is a decomposition $\Pi_1 \cup \Pi_2 = \Delta_{2,n}$ into matroid polytopes sharing a common facet $\Pi_1 \cap \Pi_2$.
- For $\Delta_{2,n}$ these look like $\sum_{j\in J} x_j = 1$ with $2 \leq |J| \leq n-2$. The common facet is a Cartesian product of simplices of dimensions $|J|-1$ and $|J^c|-1$.
- Joswig and Herrmann first systematically studied multi-splits; see also [Schroeter2017].
- **Defn.** A d-split (matroid) subdivision (of some $\Delta_{k,n}$) is a coarsest subdivision, with d maximal cells, such that these cells meet in a common cell of codimension $d-1$.
- When d is not given, simply use *multi-split*.

Blades on a hexagon

- Big picture (amplitudes): for $m_n^{(2)}$, poles are 2-splits of $\Delta_{2,n}$ and Feynman diagrams are superpositions of compatible 2-splits.
- New for $k \geq 3$ subdivisions: poles correspond to splittings of $\Delta_{k,n}$ into more than 2 chambers!
- [E,Oct2019] Introduced a new method to induce certain multi-splits:



1-split, 2-split, 3-split: induced by gluing a single *blade* $((1,2,3))$ to a vertex of a hexagon.

Compatible 2-splits of $\Delta_{2,n}$

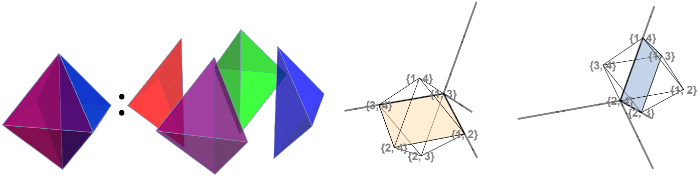
- There's a well-known compatibility rule for 2-splits of the second hypersimplex $\Delta_{2,n}$...
- Namely: Maximal cells of the subdivision of $\Delta_{2,n}$ induced by the pair of hyperplanes $\sum_{i\in J_1} x_i = 1$ and $\sum_{i\in J_2} x_i = 1$ are *matroid* polytopes if and only if at least one intersection is empty: $J_1 \cap J_2,\ J_1 \cap J_2^c,\ J_1^c \cap J_2,\ J_1^c \cap J_2^c$.
- The compatibility rule for pairs of matroid subdivisions of $\Delta_{k,n}$ involves checking a condition on each octahedral face of $\Delta_{k,n}$.

Blades induce positroidal multi-splits

- An essential question: which matroid subdivisions are induced by matroidal blade arrangements? Denote $e_{j_1,\ldots,j_k} = e_{j_1} + \cdots + e_{j_k}$. Put $\beta_J = ((1,2,\ldots,n))_{e_J}$ for the translation of the blade to the vertex e_J.
- **Theorem**[E, Oct2019] The blade $((1,2,\ldots,n))_{e_J}$ induces a multi-split positroidal subdivision of $\Delta_{k,n}$, where the maximal cells are nested matroids. The number of maximal cells in the subdivision equals the number of cyclically consecutive intervals in the labels in J.

Weakly separated collections (Non-example)

- Denote $e_J = \sum_{j \in J} e_j$ for $J \subseteq [n] = \{1, \ldots, n\}$.
- **Definition.** A pair e_{J_1}, e_{J_2} of vertices of $\Delta_{k,n}$ is *weakly separated* provided that $e_{J_1} - e_{J_2}$ does not contain the pattern $e_a - e_b + e_c - e_d$ for $a < b < c < d$ cyclically.



The non-matroidal blade arrangement of both $\{\beta_{1,3}, \beta_{2,4}\}$ subdivides the octahedron into four tetrahedra (left). But their long edge direction $e_{13} - e_{24} = e_1 - e_2 + e_3 - e_4$ fails weak separation $\Leftrightarrow$ not matroidal. Right: the two matroidal blade arrangements on the octahedron.

Enumerating weakly separated collections

The table below ([E2019]) counts maximal weakly separated collections to enumerate maximal matroidal blade arrangements on $\Delta_{k,n}$.

$n \setminus k$	2	3	4	5	6	7	8	9	10
4	2								
5	5	5							
6	14	34	14						
7	42	259	259	42					
8	132	2136	5470	2136	132				
9	429	18600	122361	122361	18600	429			
10	1430	168565	2889186	7589732	2889186	168565	1430		
11	4862	1574298	71084299			71084299	1574298	4862	
12	16796	15051702						15051702	16796

Matroidal blade arrangements (unweighted)

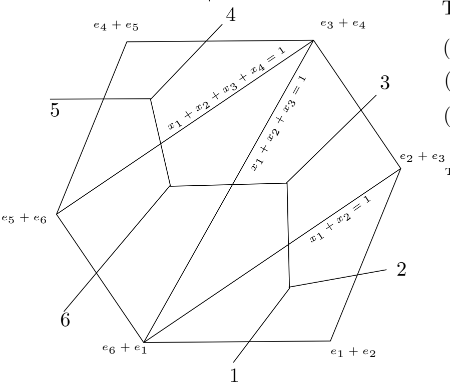
- **Theorem**[E]. An arrangement of the blade $((1, 2, \ldots, n))$ on the vertices $e_{J_1}, \ldots, e_{J_N} \in \Delta_{k,n}$ induces a matroid subdivision of $\Delta_{k,n}$ if and only if the collection $\{J_1, \ldots, J_N\}$ is *weakly separated*. Moreover, this subdivision is positroidal.

Matroidal blade arrangement on $\Delta_{2,6}$

$$\Delta_{k,n} = \{x \in [0,1]^n : \sum_{j=1}^n x_j = k\}$$

$$\Delta_{2,6} = \{x \in [0,1]^6 : x_1 + x_2 + \cdots + x_6 = 2\}$$

$$\Downarrow$$



Three 2-splits of $\Delta_{2,6}$:

(1) $x_1 + x_2 = 1$

(2) $x_1 + x_2 + x_3 = 1$

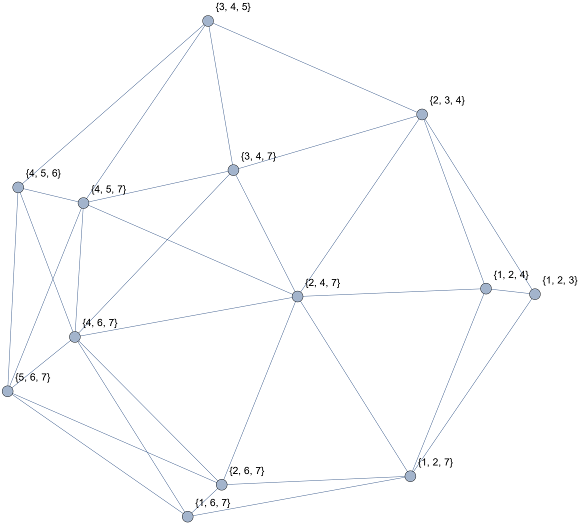
(3) $x_1 + x_2 + x_3 + x_6 = 1$

The 2-splits are pairwise compatible!

Fact: these hyperplanes divide $\Delta_{2,6}$ into four maximal cells.
These cells are polytopes s.t. their edges are parallel to roots $e_i - e_j$.
Such polytopes are called <u>matroidal</u>.

$$\beta_{26} \sim ((12_1 3456_1)), \beta_{36} \sim ((123_1 456_1)), \beta_{35} \sim ((1236_1 45_1)).$$

Matroidal Blade Arrangement on $\Delta_{3,7}$



Matroidal blade arrangement on $\Delta_{3,7}$. Vertices are connected by roots $e_i - e_j$.

Weighted Blade Arrangements: Boundary Operator

- Constructing $(\mathcal{B}(k,n),\partial)$. Boundary operator is inductive...
- $\partial_\ell(\beta_J) = \beta^{(\ell)}_{J'}$ where $J' = J \setminus \{\ell'\}$ where ℓ' is the cyclic successor of ℓ in J. Put $\partial = \sum_{j=1}^n \partial_j$.
- Frozen arrangements induce trivial subdivisions and are zero: $\beta_{i,i+1,\ldots,i+k-1} = 0$.
- Example: $\mathcal{B}(3,6)$:

$$\partial_1(\beta_{145}) = \beta^{(1)}_{45} = 0, \quad \partial_2(\beta_{145}) = \beta^{(2)}_{15}, \quad \beta_6(\beta_{145}) = \beta^{(6)}_{45} = 0,$$

$$\partial(\beta_{135}) = \beta^{(1)}_{35} + \beta^{(2)}_{15} + \beta^{(3)}_{15} + \beta^{(4)}_{13} + \beta^{(5)}_{13} + \beta^{(6)}_{35}.$$

- Example: $\mathcal{B}(4,8)$:

$$\partial_{24}(\beta_{1356}) = \beta^{(24)}_{16} \neq 0, \quad \partial_{27}(\beta_{1356}) = \beta^{(27)}_{56} = 0.$$

Weighted Matroidal Blade Arrangements

Defn.[E2020]

- A weighted blade arrangement $\beta(\mathbf{c}) = \sum_{\{i,j\}} \omega_{i,j}\beta_{i,j}$ with coefficients $\omega_{i,j} \in \mathbb{R}$ is said to be *matroidal* provided that all $\omega_{i,j} \geq 0$, and the superposition of blades $\{\beta_{i,j} : \omega_{\{i,j\}} \neq 0\}$ induces a matroid subdivision of $\Delta_{2,n}$.
- A weighted blade arrangement $\beta(\mathbf{c}) = \sum_J \omega_J \beta_J$ with coefficients $\omega_J \in \mathbb{R}$ is *matroidal* provided that for each $L \in \binom{[n]}{k-2}$, then $\partial_L(\beta(\mathbf{c}))$ is a matroidal weighted blade arrangement on $\partial_L(\Delta_{2,n}) \simeq \Delta_{2,n-(k-2)}$.

Denote by $\mathcal{Z}(k,n)$ the set of matroidal weighted blade arrangements on $\Delta_{k,n}$.

Example

- Example:

$$\beta(\mathbf{c}) = -\beta_{135} + \beta_{235} + \beta_{145} + \beta_{136}$$

then

$$\partial_1(\beta(\mathbf{c})) = -\beta^{(1)}_{35} + \beta^{(1)}_{35} + \beta^{(1)}_{36} = \beta^{(1)}_{36}.$$

- Key point: negative weights cancel on the boundary!
- In full,

$$\partial(\beta(\mathbf{c})) = \beta^{(1)}_{36} + \beta^{(2)}_{35} + \beta^{(3)}_{25} + \beta^{(4)}_{15} + \beta^{(5)}_{14} + \beta^{(6)}_{13}$$

Positive Tropical Grassmannian

- Recall: an element $\pi = \sum_J c_J e^J \in \mathbb{R}^{\binom{[n]}{k}}$ is said to be a *positive tropical Plucker vector* provided that

  $$c_{L\cup\{a,c\}} + c_{L\cup\{b,d\}} = \min(c_{L\cup\{a,b\}} + c_{L\cup\{c,d\}}, c_{L\cup\{a,d\}} + c_{L\cup\{b,c\}})$$

  for any $L \cup \{a, b, c, d\} \in \binom{[n]}{k+2}$ with $a < b < c < d$ cyclically. This set is called the *positive Dressian.*
- **Note.** Recently ([Speyer,Williams2020], [Arkani Hamed, Lam, Spradlin2020]) showed that the positive Dressian is equal to the positive tropical Grassmannian.

Wrap up: Embedding the Positive Tropical Grassmannian

- Define $\binom{[n]}{k}^{nf} = \binom{[n]}{k} \setminus \{\{i, i+1, \ldots, i+k-1\} : i = 1, \ldots, n\}$, the *nonfrozen* k-element subsets.
- Theorem [E2020]. There is an embedding of the positive tropical Grassmannian into the space of weighted matroidal blade arrangements:

$$\varphi : \sum_{J \in \binom{[n]}{k}} c_J e^J \mapsto \sum_{J \in \binom{[n]}{k}^{nf}} \omega_J \beta_J.$$

- Formula for the ω_J is an alternating sign sum over the vertices of a cube and is somewhat detailed for general (k,n); we give base case and refer to [E Dec 2019].
- *We shall characterize the image.*

From Weighted Blade Arrangements to Trop$^+$G(k,n)

Theorem [E2020]. There is an embedding of the positive tropical Grassmannian into the space of weighted matroidal blade arrangements:

$$\varphi : \sum_{J \in \binom{[n]}{k}} c_J e^J \mapsto \sum_{J \in \binom{[n]}{k}^{nf}} \omega_J \beta_J.$$

Example:

$$\sum_{\{i,j\} \in \binom{[4]}{2}} c_{ij} e^{ij} \mapsto (-c_{13}+c_{14}+c_{23}-c_{24})\beta_{13}+(-c_{24}+c_{12}+c_{34}-c_{13})\beta_{24}.$$

Notice: both coefficients $\omega_{13}, \omega_{24} \geq 0$ but at least one of them is zero

$$\Updownarrow$$

$$c_{13} + c_{24} = \min(c_{12} + c_{34}, c_{14} + c_{23})$$

These are the positive tropical Plucker relations!

Example

1. Example: we look at two matroidal weighted blade arrangements and then add them to get a third.

$$\sum_{j=1}^{6} \partial_j(-\beta_{135} + \beta_{235} + \beta_{145} + \beta_{136})$$

$$= \beta_{36}^{(1)} + \beta_{35}^{(2)} + \beta_{25}^{(3)} + \beta_{15}^{(4)} + \beta_{14}^{(5)} + \beta_{13}^{(6)}$$

$$\sum_{j=1}^{6} \partial_j(\beta_{236}) = \beta_{36}^{(1)} + \beta_{36}^{(2)} + \beta_{26}^{(3)}$$

2. Comparing boundaries term by term $\Rightarrow$ that for all $a, b \geq 0$,

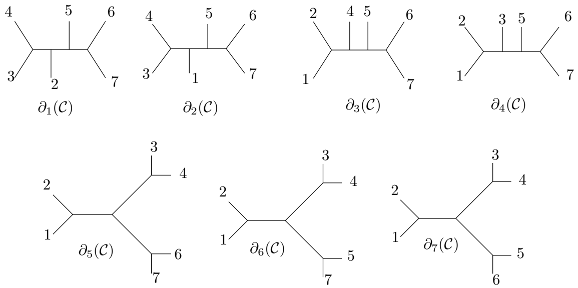
$$a\beta_{236} + b(-\beta_{135} + \beta_{235} + \beta_{145} + \beta_{136})$$

is a matroidal weighted blade arrangement.

3. In fact, not hard to identify this with (the image of) a 2-dimensional cone in $\mathrm{Trop}^+ G(3,6)$.

Generalized Feynman Diagrams

- *Generalized Feynman Diagrams.* [Borges-Cachazo2019]
- Calculated maximal cones [CGUZ 2019] ($\Rightarrow$ CEGM amplitudes) for $\text{Trop}^+ G(k,n)$ for $(k,n) \in \{(3,6),(3,7),(3,8),(3,9),(4,8),(3,9)\}$.



Example (above): GFD's on faces $x_j = 1$ of $\Delta_{3,7}$ by the matroidal blade arrangement $\mathcal{C} = \{\beta_{124}, \beta_{247}, \beta_{267}, \beta_{347}, \beta_{457}, \beta_{467}\}$.

$$\partial_1(\mathcal{C}) = \{\beta_{2,4}, \beta_{4,7}, \beta_{5,7}\}.$$

Back to blades; towards the kinematic space

Let $V_0^n \subset \mathbb{R}^n$ be the hyperplane $x_1 + \cdots + x_n = 0$.

Defn. Let $h : V_0^n \to \mathbb{R}$ be the piece-wise linear function $h(x) = \min\{L_1(x), \ldots, L_n(x)\}$, where

$$L_j = x_{j+1} + 2x_{j+2} + \cdots (n-1)x_{j-1}.$$

Prop.[E,Oct2019]. The blade $((1, 2, \ldots, n))$ equals the bend locus of the function $h(x)$. That is,

$$((1, 2, \ldots, n)) = \{x \in V_0^n : (L_i(x) = L_j(x)) \leq L_\ell(x) \text{ for all } \ell \neq i, j\}.$$

Height functions and planar basis

Defn.[E,Dec 2019]. At each vertex $e_J(=\sum_{j\in J} e_j) \in \Delta_{k,n}$, we'll glue a copy of $((1,2,\ldots,n))$ and define a linear form on $\mathcal{K}_{k,n}$: set

$$\rho_J(x) = h(x-e_J), \text{ and } \quad \eta_J = -\frac{1}{n}\sum_{e_I\in\Delta_{k,n}} \rho_J(e_I)s_I.$$

Thm. [E, 2020]. The set $\{\eta_J : J \text{ is nonfrozen}\}$ is a basis[1], the *planar basis*, for the space of linear functions on the kinematic space $\mathcal{K}_{k,n}$.

These functions η_J are highly combinatorially structured; we discuss some aspects now...

[1] *frozen* elements are zero: $\eta_{i,i+1,\ldots,i+(k-1)} = 0$

Planar kinematic invariants η_J: basics

Warm up, $k = 2$. On the kinematic space $\mathcal{K}_{2,6}$

$$\begin{aligned}
\eta_{24} &= \frac{1}{4}\left(3s_{12} + 2s_{13} + s_{14} + s_{23} + 3s_{34}\right) \\
&\equiv s_{34} \\
\eta_{25} &= s_{34} + s_{35} + s_{45} \\
\eta_{23} &= \frac{1}{4}\left(2s_{12} + s_{13} + 4s_{14} + 3s_{24} + 2s_{34}\right) \\
&\equiv 0.
\end{aligned}$$

Of course this all works beautifully for $k \geq 3$: e.g., (3,6):

$$\begin{aligned}
\eta_{135} &= \frac{1}{6}\left(3s_{123} + 2s_{124} + s_{125} + 6s_{126} + \cdots + s_{356} + 6s_{456}\right) \\
&\equiv s_{123} + s_{126} + s_{136} + s_{234} + s_{235} + s_{236}.
\end{aligned}$$

This is one of the new poles ("$R_{16,23,45}$") in $m^{(3)}(\mathbb{I}_6, \mathbb{I}_6)$!

Inverse transformation

- Nice "cubical" rule for expanding s_J as a sum of η_J's ($k=2$ case familiar):

$$s_{25} = -(\eta_{14} - \eta_{15} - \eta_{24} + \eta_{25}).$$

- There is a generalization to $k \geq 3$:

$$\begin{aligned} -s_{235} &= \eta_{235} - \eta_{234} - \eta_{135} + \eta_{134} \\ -s_{246} &= \eta_{246} - \eta_{146} - \eta_{236} + \eta_{136} - \eta_{245} + \eta_{145} + \eta_{235} - \eta_{135}. \end{aligned}$$

Planar basis: explicit inversion formula

- **Prop.** [E,Dec2019] Given a nonfrozen vertex $e_J \in \Delta_{k,n}$ s.t. J has $t(\geq 2)$ cyclic intervals, with cyclic initial points say $j_1, \ldots, j_t$, consider the t-dimensional cube

$$C_J = \left\{ J_L = \{j_1 - \ell_1, \ldots, j_t - \ell_t\} : L = (\ell_1, \ldots, \ell_t) \in \{0,1\}^t \right\}.$$

Then the following "cubical" relation among linear functionals holds identically on $\mathbb{R}^{\binom{n}{k}}$, as well as on the subspace $\mathcal{K}_{k,n}$:

$$\sum_{L \in C_J} (-1)^{L \cdot L} \eta_{J_L} = -s_J,$$

where $L \cdot L$ is the number of 1's in the 0/1 vector L.

Complementary Story: Rational Functions on Projective Space

For any $1 < i < j \leq n$, define a linear function $\delta_{i,j} = \sum_{t=i-1}^{j-2} x_t$ on $\mathbb{C}^{n-2}$. Put $\delta_{1,j} = 1$.
Define rational functions $u_{i,j}$ whenever i, j are not cyclically adjacent in $\{1, \ldots, n\}$:

$$u_{i,n} = \frac{\delta_{i+1,n}}{\delta_{i,n}}, \quad \text{otherwise } u_{i,j} = \frac{\delta_{i+1,j}\delta_{i,j+1}}{\delta_{i,j}\delta_{i+1,j+1}}.$$

Example, $n = 5$.

$$u_{2,4} = \frac{\delta_{3,4}\delta_{2,5}}{\delta_{2,4}\delta_{3,5}} = \frac{x_2(x_1 + x_2 + x_3)}{(x_1 + x_2)(x_2 + x_3)}, \quad u_{2,5} = \frac{\delta_{3,5}}{\delta_{2,5}} = \frac{x_2 + x_3}{x_1 + x_2 + x_3}.$$

(1) **Note:** $u_{i,j}$ are well-defined on $\mathbb{CP}^{n-3}$.
(2) We'll see that these are specializations of certain cross-ratios defined on the Riemann sphere.

Binary structures for the functions $u_{i,j}$

We have the *binary property*, well-known (in case $k = 2$):

- **Prop.** If some $u_{i,j}(g) = 0$, then $u_{k,\ell}(g) = 1$ whenever the pair $\{\{i,j\},\{k,\ell\}\}$ is crossing. E.g. $n = 4$: $u_{1,3} + u_{2,4} = 1$.
- The *binary property* follows from a known stronger *binary identity*:

$$u_{i,j} = 1 - \prod_{((i,j),(k,\ell))\ \text{crossing}} u_{k,\ell} \tag{1}$$

Example (1) $n = 4$:

$$u_{13} = 1 - u_{24} \text{ where } u_{13} = \frac{x_1}{x_1 + x_2},\ u_{24} = \frac{x_2}{x_1 + x_2}.$$

Example (2) $n = 6$:

$$u_{24} = 1 - u_{13} u_{35} u_{36}.$$

Planar Cross-ratios on projective configurations in $\mathbb{CP}^{k-1}$

Claim: there exists a system of projective invariants on $G(k,n)/(\mathbb{C}^*)^n$ satisfying the binary property, but where the binary identities are *rational*.

Let $J \in \binom{[n]}{k}^{nf}$. Define cubes in $\Delta_{k,n}$ by

$$\mathcal{U}(J) = \{e_J\} \cup \{e_J + e_{j+1} - e_j : (j, j+1) \in J \times J^c\},$$

$$\mathcal{D}(J) = \{e_J\} \cup \{e_J + e_{j-1} - e_j : (j-1, j) \in J^c \times J\}.$$

$$w_J = \prod_{M \in \mathcal{U}(J)} p_M^{k - \#(M \cap J) + 1},$$

where p_J is the minor with column set J of a given $k \times n$ matrix. Here indices on e_J are cyclic with period n. For 3 and $n = 6$, one has e.g.

$$w_{136} = \frac{p_{146} p_{236}}{p_{136} p_{246}}, \quad w_{135} = \frac{p_{136} p_{145} p_{235} p_{246}}{p_{135} p_{146} p_{236} p_{245}}.$$

Binary Property

- **Thm**. [E2021: In Prep]. If we have $w_I = 0$, then $w_J = 1$ for any $J \in \binom{[n]}{k}$ such that (I, J) is not weakly separated.
- Proof is inductive on k, starting from $k = 2$, and uses a certain change of basis of $\mathbb{R}^{\binom{n}{k}}$ coming from blade arrangements, inspired by work on amplitudes [E2019].

Example. $(k, n) = (2, 5)$. Claim: $w_{24} = 0 \Rightarrow w_{13} = 0$. Cancellations and a Plucker identity give

$$w_{1,3} + w_{2,4}w_{2,5} = \frac{p_{23}p_{14}}{p_{13}p_{24}} + \left(\frac{p_{34}p_{25}}{p_{24}p_{35}}\right)\left(\frac{p_{35}p_{21}}{p_{25}p_{31}}\right) \quad (2)$$

$$= \frac{p_{23}p_{14} + p_{12}p_{34}}{p_{13}p_{24}} = 1. \quad (3)$$

Example. Claim: $w_{135} = 0 \Rightarrow w_{246} = 1$. Can show that

$$w_{246} = \frac{1 - w_{135}w_{235}w_{136}w_{356}}{1 - w_{135}w_{235}w_{136}w_{236}}.$$

Connection to u_{ij}'s: cross-ratios on point configurations in $\mathbb{CP}^1$

Start with the map $\mathbb{C}^{n-2} \hookrightarrow G(2,n)/(\mathbb{C}^*)^n$,

$$(x_1, x_2, \ldots, x_n) \mapsto g = \begin{pmatrix} 1 & 0 & x_1 & x_1+x_2 & x_1+x_2+x_3 & \cdots \\ 0 & 1 & 1 & 1 & 1 & \end{pmatrix}.$$

Clearly this induces an embedding $\mathbb{CP}^{n-3} \hookrightarrow G(2,n)/(\mathbb{C}^*)^n$.
Claim: the rational functions $u_{i,j}$ factor through $G(2,n)/(\mathbb{C}^*)^n$, so that

$$w_{i,j} = u_{i,j}.$$

For example (noting signs cancel),

$$w_{24} = \frac{p_{34}p_{25}}{p_{24}p_{35}} = \frac{x_2(x_1+x_2+x_3)}{(x_1+x_2)(x_2+x_3)} = u_{24}.$$

Small Question: is there a similarly nice binary story for $G(k,n)/(\mathbb{C}^*)^n$? Yes! See [E2021 in prep].

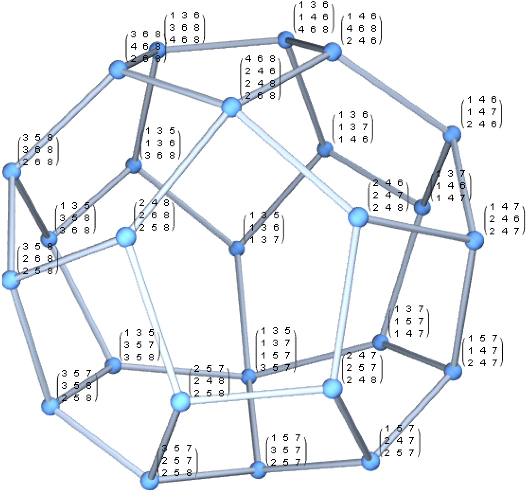
Kinematic space:
$\mathcal{K}_{2,n} = \left\{(s_{i,j}) \in \mathbb{R}^{\binom{n}{2}} : \sum_{j\neq i} s_{i,j} = 0 \quad i = 1, \ldots, n\right\}$
Construction of [E2019], specialized to $k = 2$.
Denote $L_j(x) = x_{j+1} + 2x_{j+2} + \cdots + (n-1)x_{j-1}$.
Define

$$\eta_{i,j} \quad := \quad -\frac{1}{n} \sum_{1\leq a<b\leq n} \min\{L_t(e_a + e_b - e_i - e_j) : t = 1, \ldots, n\} s_{a,b}$$

Claim. We have

$$\sum_{1\leq i<j\leq n} \log(\det(g_i, g_j))s_{i,j} \quad = \quad \sum_{(a,b)\in\binom{[n]}{2}\setminus\{(i,i+1)\}} \log(u_{a,b})\eta_{a,b}$$

Proof. First part, use $-s_{i,j} = \eta_{i,j} - \eta_{i-1,j} - \eta_{i,j-1} + \eta_{i-1,j-1}$.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Thank you!

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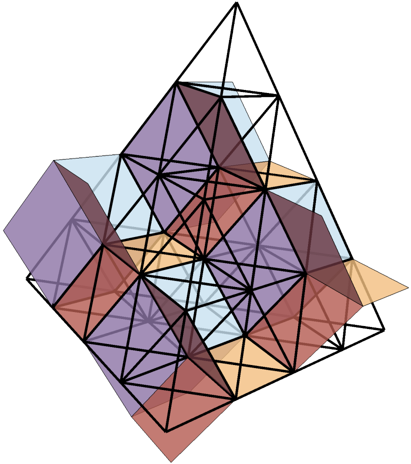
Blade Arrangements in Four Coordinates



Four blades arranged in a dilated tetrahedron (truncated to improve clarity). Important: every octahedron is subdivided at most once!

$m^{(3)}(\mathbb{I}_6, \mathbb{I}_6)$ in the planar basis

- In the planar basis, $m^{(3)}(\mathbb{I}_6, \mathbb{I}_6)$ has the expression

$$\begin{aligned} m^{(3)}(\mathbb{I}_6, \mathbb{I}_6) &= \frac{1}{\eta_{125}\eta_{134}\eta_{135}\eta_{145}} + \frac{1}{\eta_{124}\eta_{125}\eta_{134}\eta_{145}} \\ &+ \frac{1}{\eta_{136}\eta_{145}\eta_{146}\left(-\eta_{135}+\eta_{136}+\eta_{145}+\eta_{235}\right)} \\ &+ \frac{\eta_{136}+\eta_{145}+\eta_{235}}{\eta_{135}\eta_{136}\eta_{145}\eta_{235}\left(-\eta_{135}+\eta_{136}+\eta_{145}+\eta_{235}\right)} + 44 \text{ more.} \end{aligned}$$